The moduli space of germs of generic families of analytic diffeomorphisms unfolding of a codimension one resonant diffeomorphism or resonant saddle*

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Abstract

In this paper we describe the moduli space of germs of generic analytic families of complex 1-dimensional resonant analytic diffeomorphisms of codimension 1. In [13], it was shown that the Ecalle modulus can be unfolded to give a complete modulus for such germs. As the function of the canonical parameter, the modulus is defined on two sectors giving a covering of a neighborhood of the origin in the parameter space. As in the case of the Ecalle modulus, the modulus is defined up to a linear scaling depending only on the parameter.

The *compatibility condition* is obtained by considering the region of intersection of the two sectors in parameter space, which we call the Glutsyuk sectors. There, both the fixed point and the periodic orbit are hyperbolic and they are connected by the orbits of the diffeomorphism. This yields an alternate description of the equivalence class by the Glutsyuk modulus: near each of the fixed point and of the periodic orbit we construct a change of coordinate to the normal form. The Glutsyuk modulus measures the obstruction of having one being the analytic extension of the other. In the intersection of the two sectors, we have two representatives of the modulus which describe the same dynamics. A necessary compatibility condition is that they have the same Glutsyuk modulus. This necessary condition becomes sufficient for realizability

The compatibility condition implies the existence of a linear scaling for which the modulus is 1-summable in ϵ, whose directions of non-summability coincide with the direction of real multipliers at the fixed point and periodic orbit. Conversely, we show that the compatibility condition (which implies the summability property) is sufficient to realize the modulus as coming from an analytic unfolding, thus giving a complete description of the space of moduli.

1 Introduction

The paper [13] presented a complete modulus of analytic classification for a germ of generic 1-parameter family unfolding of a codimension 1 resonant diffeomorphism and its application to the orbital analytic classification of a germ of generic 1-parameter family unfolding of a codimension 1 resonant saddle of a 2-dimensional vector field. It was shown that this modulus

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was an unfolding of the Ecalle modulus. At the time, the moduli space was out of reach. The present paper fills this hole.

The problem studied here is part of a large class of problems trying to understand the structure of the singularities of dynamical systems:

1. When are two germs of diffeomorphisms conjugate?
2. When are two germs of vectors fields orbitally equivalent?
3. When are two germs of families of diffeomorphisms conjugate?
4. When are two germs of families of vectors fields orbitally equivalent?

While these problems are simple to state, their solution is quite involved. The first two problems were solved by Ecalle for 1-dimensional resonant diffeomorphisms (Voronin also studied the case of multiplier 1), and Martinet-Ramis for the 2-dimensional resonant saddle. In both cases, the "object" was classified by some formal invariants and a very complicated analytic invariant of functional type. The paper [7] answered the third problem for the case of a codimension 1 parabolic point with multiplier equal to 1: this shed a new light on the first two problems: by studying the unfolding, it explained why the invariant was so complex: the invariant encodes some of the dynamics of the unfolded object. The paper [13] treated the third and fourth problem when the family unfolded a fixed point with resonant multiplier $\exp\left(2\pi i\frac{p}{q}\right)$, or a resonant saddle. More recently, the paper [1] describes the moduli space for germs of families of diffeomorphisms unfolding a parabolic fixed point, thus ending the study started in [7]. We address the same problem here for generic unfoldings of codimension 1 resonant diffeomorphisms or resonant saddles. The resonant multiplier $\exp\left(2\pi i\frac{p}{q}\right)$ expresses that the fixed point is multiple: it has merged with a periodic orbit of period q. When $q > 1$, this yields additional technical difficulties compared to the case $q = 1$. Indeed, in [13], it is shown that two such families of diffeomorphisms are conjugate if and only if their q-th powers are conjugate and we work with their q-th powers. Here fortunately, working simultaneously on moduli spaces for germs of families of analytic vector fields will allow to simplify the realization problem and to reduce it to the case $q = 1$.

To do this, we first prove that any germ of family unfolding a resonant diffeomorphism can be realized as the holonomy map of a separatrix of a germ of family of vector fields. The family of holonomies of the second separatrix is, in turn, a second germ of family of diffeomorphisms. Iterating this trick a finite number of times allows to relate any family unfolding a resonant diffeomorphism to a family unfolding a fixed point with multiplier equal to 1.

The next step is to study the relationship between the invariants associated to the holonomies of the two separatrices in a germ of family unfolding a resonant saddle. The tool for this is the Dulac map. While the formal invariants and canonical parameters are not the same, the "nonlinear analytic part" of the modulus is the same (as noticed by Martinet and Ramis in the non-unfolded case). To do this, we work intensively with the Martinet-Ramis point of view, which consists in describing the orbit space of the diffeomorphism by two spheres with the neighborhoods of 0 and ∞ identified. We extend this point of view to the unfolding. Exact formulae showing the relationship between the formal and analytic part of the invariants of the holonomies of the two separatrices are given.

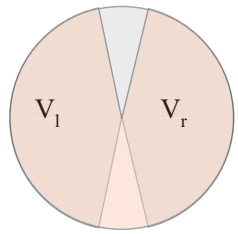


Figure 1: The two small sectors V_l and V_r.

Finally, we identify the moduli space. As explained before, it suffices to consider the case $q = 1$, and hence germs of families of diffeomorphisms in prepared form

$$f_\epsilon(z) = z + z(z-\epsilon)(1 + B(\epsilon) + A(\epsilon)z + O(z(z-\epsilon)),$$

so that $f'_\epsilon(0) = \exp(-\epsilon)$, thus guaranteing that the (canonical) parameter ϵ be an analytic invariant. The modulus of a germ of family of diffeomorphisms is given by

- the formal invariant $a(\epsilon)$ which depends analytically on ϵ,
- two families of germs of analytic diffeomorphisms $(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})\big|_{\epsilon\in V_\pm}$, where $V_\pm$ are two sectors whose union is a punctured neighborhood of the origin in ϵ-space and $\boldsymbol{\psi}^0_{\epsilon,\pm} \in (\mathbb{C}, 0)$, $\boldsymbol{\psi}^\infty_{\epsilon,\pm} \in (\mathbb{C}, \infty)$. These families of germs are defined up to linear changes of coordinates on $\mathbb{CP}^1$ depending analytically on ϵ.

The two sectors $V_\pm$ intersect in two smaller sectors (Figure 1): over these, we have two different descriptions of the dynamics. Conversely, given two families $(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})\big|_{\epsilon\in V_\pm}$, the sufficient condition for realizability consists in expressing that they encode conjugate dynamics over the intersection sectors. This, in turn, implies that for an adequate choice of linear coordinates on $\mathbb{CP}^1$, then $\boldsymbol{\psi}^0_\epsilon$ and $\boldsymbol{\psi}^\infty_\epsilon$ are 1-summable in ϵ. The realization is then done in two steps:

- the local realization in z, yielding two families, one for each sector $V_\pm$;
- the global realization, where we correct to a uniform family in ϵ.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 describes the modulus of analytic classification. Section 4 gives the compatibility condition. Section 5 describes the link between the holonomies of the two separatrices of a resonant saddle. In Section 6, it is proved that any generic unfolding of a codimension 1 diffeomorphism can be realized as the holonomy of a separatrix of an unfolding of a resonant saddle. In Section 7, the realization theorem is proved simultaneously for unfoldings of resonant diffeomorphisms and of resonant saddles. Section 8 studies the particular case of an unfolding of a resonant real vector field.

2 Preliminaries

2.1 Notations

The notations collected here are often referred to in the paper.

- L_C: the linear map
$$L_C(w) = Cw; \tag{2.1}$$
- T_B: the translation
$$T_B(W) = W + B; \tag{2.2}$$
- the q-root of unity
$$\tau = \exp\left(\frac{2\pi i}{q}\right); \tag{2.3}$$
- E: the map
$$E(W) = \exp(-2\pi i W); \tag{2.4}$$
with inverse $E^{-1}(w) = -\frac{1}{2\pi i}\ln(w)$;
- Σ: the map corresponding to complex conjugation
$$\Sigma(w) = \overline{w}; \tag{2.5}$$
- Ξ: the reflection with respect to the imaginary axis
$$\Xi(w) = -\overline{w}. \tag{2.6}$$

2.2 Preparation of the family

We recall briefly the results of [13]. We consider a germ of generic resonant diffeomorphism of the form
$$f_0(z) = e^{\frac{2i\pi p}{q}} z + \frac{e^{\frac{2i\pi p}{q}}}{q} z^{q+1} + o(z^{q+1}). \tag{2.7}$$
Then f_0^q has a fixed point at the origin of multiplicity $q+1$, which corresponds, for f_0, to the coalescence of a fixed point with a periodic orbit of period q: the fixed point and periodic orbit bifurcate in a generic unfolding. Because we can always localize the fixed point at the origin, bring the family in normal form up to order $q+1$ and rescale, then a germ of generic unfolding can be taken of the form
$$f_\epsilon(z) = \left(e^{\frac{2i\pi p}{q}} - \alpha\right) z + \frac{e^{\frac{2i\pi p}{q}}}{q} z^{q+1} + o(z^{q+1}) \tag{2.8}$$
with α a small parameter.

It is proved in [13] that we can limit ourselves to consider the conjugacy problem for the q-th iterate $g_\epsilon = f_\epsilon^q$ of f_ϵ, which has the form
$$g_\epsilon(z) = z(1-\epsilon) + (1 + O(\epsilon))z^{q+1} + o(z^{q+1}), \tag{2.9}$$
where
$$(1-\epsilon) = \left(e^{\frac{2\pi i p}{q}} - \alpha\right)^q. \tag{2.10}$$

Because g_0 has multiplier equal to 1 at the origin (and hence fixed points instead of periodic points), it is easier to work with $g_\epsilon = f_\epsilon^q$ than with f_ϵ. The following proposition is shown in [13].

Proposition 2.1 *For a family of the form* (2.9)*, there exists an analytic reparameterization of* (z, ϵ) *tangent to the identity and fibered over the parameter space allowing to reduce the study to that of a family in prepared form*

$$g_\epsilon(z) = z + z(z^q - \epsilon)[1 + B(\epsilon) + A(\epsilon)z^q + z(z^q - \epsilon)h(\epsilon, z)], \tag{2.11}$$

where $B(\epsilon) = (1 - \epsilon - \exp(-\epsilon))/\epsilon = O(\epsilon)$*, and* $h(0, 0) = 0$*. The diffeomorphism* g_ϵ *has fixed points* $z_0 = 0$ *and* z_j*,* $j = 1, \ldots, q$ *with* $z_j^q = \epsilon$*. The multiplier* λ_0 *of the fixed point* $z_0 = 0$ *satisfies*

$$\lambda_0 = \exp(-\epsilon), \tag{2.12}$$

and hence the parameter ϵ *is an analytic invariant for* g_ϵ*, which is called the* canonical parameter*. Let* $\lambda_1, \ldots \lambda_q$*, be the multipliers of the fixed points* z_j*, where* $z_j^q = \epsilon$*. The formal parameter*

$$a(\epsilon) = \frac{1}{\ln \lambda_0} + \sum_{j=1}^{q} \frac{1}{\ln \lambda_j} \tag{2.13}$$

depends analytically on ϵ*. It is a formal invariant of* g_ϵ*.*

2.3 Strategy

Considering a prepared family (2.11), the strategy is to construct Fatou coordinates: these are changes of coordinates which transform the family (2.11) to the associated "model family" which is the time-one map of the vector field

$$\frac{z(z^q - \epsilon)}{1 + a(\epsilon)z^q} \frac{\partial}{\partial z}, \tag{2.14}$$

obtained as follows:

- the fixed points $z_0, z_1, \ldots, z_q$ of g_ϵ coincide with the singular points of (2.14);
- let μ_j be the eigenvalue of (2.14) at the singular point z_j and λ_j be the multiplier of g_ϵ at z_j. Then $\lambda_j = \exp(\mu_j)$;
- Since the eigenvalues at the singular points $z_1, \ldots, z_q$ of (2.14) have the form

$$\mu_0 = -\epsilon, \qquad \mu_j = \frac{q\epsilon}{1 + a(\epsilon)\epsilon}, \quad j = 1, \ldots, q, \tag{2.15}$$

  then $\lambda_j = \exp(\mu_j)$ forces the choice of $a(\epsilon)$.

The construction of Fatou coordinates is given in [13] and we recall the essential step.

2.4 The two charts

We study the dynamics of the germ of family $g_\epsilon(z)$ on any sufficiently small neighborhood of the origin in z-coordinate which we can choose of the form $U = \{z, |z| < r\}$ with $r \in (0, 1)$ for all sufficiently small values of the parameter ϵ in a small ball $V = \{\epsilon; |\epsilon| < \rho\}$. We limit

ourselves to values of ϵ sufficiently small so that the fixed points of g_ϵ remain inside U. For this it suffices to take

$$\rho < \frac{r^q}{2}, \tag{2.16}$$

a condition which will be assumed throughout the paper.

We need to cover V with two sectors, each of opening $\pi + 2\delta$ with $\delta \in (0, \frac{\pi}{2})$. The parameter $\delta \in (0, \frac{\pi}{2})$ is chosen at the beginning and kept fixed for all the treatment. We give a uniform treatment of g_ϵ over the two following two sectors of V:

$$\begin{array}{rcl} V_{\delta,+} & = & \{\epsilon \in V \,|\, \arg \epsilon \in (-\delta, \pi + \delta)\}, \\ V_{\delta,-} & = & \{\epsilon \in V \,|\, \arg \epsilon \in (\pi - \delta, 2\pi + \delta)\}. \end{array} \tag{2.17}$$

2.5 The lifted diffeomorphism

We first introduce a change of coordinate which nearly rectifies the family g_ϵ to the translation by 1 and sends the fixed points to infinity. We will in particular consider the translation $T_{\alpha(\epsilon)}(Z) = Z + \alpha(\epsilon)$ with

$$\alpha(\epsilon) = \begin{cases} \frac{2\pi i}{q\epsilon}, & \epsilon \neq 0, \\ 0, & \epsilon = 0. \end{cases} \tag{2.18}$$

We introduce the change of coordinate $p_\epsilon : S_\epsilon \to \mathbb{CP}^1 \setminus \{0, z_1, \ldots, z_q\}$ given by

$$z = p_\epsilon(Z) = \begin{cases} \left(\frac{\epsilon}{1-e^{q\epsilon Z}}\right)^{1/q}, & \epsilon \neq 0, \\ \left(-\frac{1}{qZ}\right)^{1/q}, & \epsilon = 0, \end{cases} \tag{2.19}$$

where S_ϵ is the Riemann surface of the function

$$\begin{cases} \left(\frac{e^{q\epsilon Z}-1}{\epsilon}\right)^{1/q}, & \epsilon \neq 0, \\ Z^{1/q}, & \epsilon = 0. \end{cases} \tag{2.20}$$

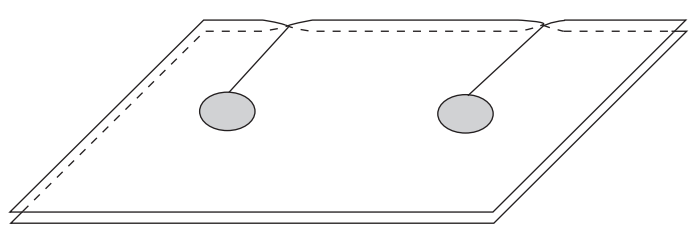


Figure 2: The domain of Z in the case $q = 2$

For $\epsilon \neq 0$ it is univalued when the image is restricted to a strip of width $\alpha(\epsilon)$. We can lift the map $T_{\alpha(\epsilon)}$ to S_ϵ.

The image of $U \setminus \{0, z_1, \ldots, z_q\}$ under p_ϵ^{-1} is

$$\hat{U}_\epsilon = S_\epsilon \setminus \cup_{j\in\mathbb{Z}} B_j, \tag{2.21}$$

where B_0 is the component of $p_\epsilon^{-1}(\mathbb{C} \setminus U)$ which contains the origin and $B_i = T^i_{\alpha(\epsilon)}(B_0) = T_{i\alpha(\epsilon)}(B_0)$. B_0 is called the *fundamental hole*. It is a q-covering of a neighborhood of the origin.

We lift the function $g_\epsilon(z)$ to a function $G_\epsilon(Z)$ commuting with $T_{q\alpha(\epsilon)}$.

2.6 Translation domains

The Fatou coordinates are defined on maximal domains in Z-space called translation domains.

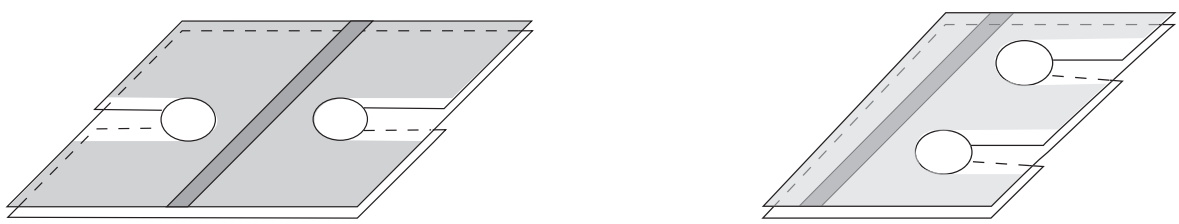
Definition 2.2

1. A line $\ell \subset \hat{U}_\epsilon$ is called an *admissible line* if ℓ and $G_\epsilon(\ell)$ are disjoint and the strip $\hat{C}_\epsilon(\ell)$ between ℓ and $G(\ell)$ is included in $\hat{U}_\epsilon$. The strip $\hat{C}_\epsilon(\ell)$ is called an *admissible strip*.

2. Let ℓ be an admissible line for G_ϵ. The *translation domain* associated with ℓ is the set
$$Q_\epsilon(\ell) = \{Z \in \hat{U}_\epsilon | \exists n \in \mathbb{Z}\ G^n_\epsilon(Z) \in \hat{C}_\epsilon(\ell) \quad \text{and} \quad \forall j \in [0,n] \subset \mathbb{Z}, G^j_\epsilon(Z) \in \hat{U}_\epsilon\}. \tag{2.22}$$
(For $n < 0$, $[0,n] = \{j \in \mathbb{Z} | n \leq j \leq 0.$)

3. A *Lavaurs translation domain* (Figure 3(a)) is a domain associated with an admissible line passing between the fundamental hole and one of its two adjacent holes (notation Q^L_ϵ).

4. A *Glutsyuk translation domain* (Figure 3(b)) is a domain associated with an admissible line parallel to the line of holes (notation Q^G_ϵ).



(a) Lavaurs translation domain (b) Glutsyuk translation domain

Figure 3: A fundamental domain $\hat{C}_\epsilon(\ell)$ associated to an admissible line ℓ and the translation domain it generates (the figure is drawn for $q = 2$)

2.7 Fatou coordinates

Theorem 2.3 *Let $Q_\epsilon = Q_\epsilon(\ell)$ be any translation domain.*

1. *There exists a holomorphic diffeomorphism $\Phi_\epsilon : Q_\epsilon \to \mathbb{C}$, such that*
$$\Phi_\epsilon(G_\epsilon(Z)) = \Phi_\epsilon(Z) + 1, \tag{2.23}$$
for $Z \in Q_\epsilon \cap G_\epsilon^{-1}(Q_\epsilon)$. Moreover,
$$\lim_{Im(Z)\to\pm\infty} Im(\Phi_\epsilon(Z)) = \pm\infty. \tag{2.24}$$

2. *If $\Phi_{1,\epsilon}$ and $\Phi_{2,\epsilon}$ are two solutions of* (2.23)*, then there exists $A \in \mathbb{C}$ such that $\Phi_{2,\epsilon}(Z) = A + \Phi_{1,\epsilon}(Z)$. In particular, if $Z_0(\epsilon) \in Q_\epsilon(\ell)$, there exists a unique holomorphic diffeomorphism Φ_ϵ satisfying* (2.23) *together with $\Phi_\epsilon(Z_0(\epsilon)) = 0$.*

Moreover, let $Z_0(\epsilon) \in Q_\epsilon$ depend holomorphically on ϵ (including at $\epsilon = 0$) and let Φ_ϵ be the Fatou coordinate defined on Q_ϵ for $\epsilon \in V_{\delta,\pm}$ and normalized by $\Phi_\epsilon(Z_0(\epsilon)) = 0$.

Let

$$Q_\pm = \cup_{\epsilon \in V_{\delta,\pm}} (\{\epsilon\} \times Q_\epsilon) \subset \mathbb{C}^2 \tag{2.25}$$

and let $\Phi_\pm : Q_\pm \to \mathbb{C}$ defined by $\Phi_\pm(\epsilon, Z) = \Phi_\epsilon(Z)$. The function $\Phi_\pm$ is holomorphic in $Int(Q_\pm)$ (i.e. for $\epsilon \neq 0$), and continuous in $Q_\pm$.

Definition 2.4 A function Φ_ϵ constructed in Theorem 2.3 is called a *Fatou coordinate associated with the translation domain Q_ϵ*. The base point of a Fatou coordinate is the point $Z_0(\epsilon) = \Phi_\epsilon^{-1}(0)$.

3 The modulus of analytic classification

For $\epsilon = 0$, Fatou coordinates are defined on translation domains which belong to the complement of a q-sheeted neighborhood of 0. If we consider an admissible line located in a sheet on one side of the hole and the translation domain it generates, then, for $q \geq 2$, this domain intersects exactly two translation domains associated to admissible lines located on the other side of the hole B_0 (see Figure 4). Moreover, each of the two intersections is simply con-

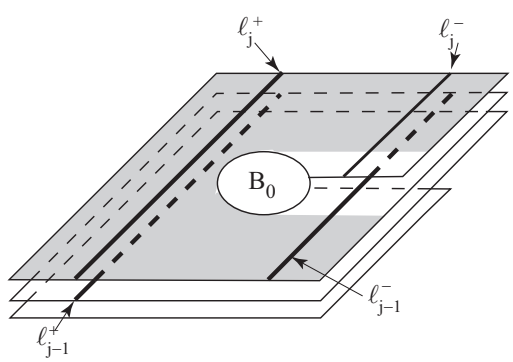


Figure 4: Four admissible lines and one translation domain (here $q = 3$)

nected, yielding that a comparison of the two Fatou coordinates is possible only in a domain containing a half-plane. When $\epsilon \neq 0$, we have a similar picture, but repeated at each of the holes. Remember that the whole surface looks like Figure 2.

So, for the sector $V_{\delta,+}$ (resp. $V_{\delta,-}$), we consider $2q$ global Fatou coordinates $\Phi_{j,+}^\pm$ (resp. $\Phi_{j,-}^\pm$) generated by admissible lines $\ell_{j,+}^\pm(\epsilon)$ (resp. $\ell_{j,-}^\pm(\epsilon)$), $j = 1, \ldots q$, located respectively between B_0 and either B_1 or B_{-1} on the different sheets and generating admissible strips $\hat{C}_{j,\epsilon,+}^\pm$ (resp. $\hat{C}_{j,\epsilon,-}^\pm$). The lines $\ell_{j,-}^-$ and $\ell_{j,+}^+$ (resp. $\ell_{j,+}^-$ and $\ell_{j,-}^+$) pass through B_0 and B_{-1} (resp. B_0 and B_1). (For the index j, we work (mod q).) They generate translation domains $Q_{j,\epsilon,\pm}^\pm$. Their indices are chosen so that the translation domains of $\ell_{j,\pm}^+(\epsilon)$ and $\ell_{j,\pm}^-(\epsilon)$ (resp.

$\ell^+_{j+1,\pm}(\epsilon)$ and $\ell^-_{j,\pm}(\epsilon)$) intersect and contain an "upper domain" (resp. "lower domain"), i.e. a domain whose intersection with $\hat{C}^{\pm}_{j,\epsilon,\pm}$ contains an upper end (resp. lower end) of the cylinder $\hat{C}^{\pm}_{j,\epsilon,\pm}/G_\epsilon$.

We define

$$\begin{cases} \Psi^{\infty}_{j,\epsilon,\pm} = \Phi^-_{j,\epsilon,\pm} \circ (\Phi^+_{j,\epsilon,\pm})^{-1}, \\ \Psi^{0}_{j,\epsilon,\pm} = \Phi^-_{j,\epsilon,\pm} \circ (\Phi^+_{j+1,\epsilon,\pm})^{-1}, \end{cases} \tag{3.1}$$

$j = 1, \dots, q$, where we identify $\Phi^+_{q+1,\epsilon,\pm} = \Phi^+_{1,\epsilon,\pm}$.

Remark 3.1 Note that the Fatou coordinates are only defined up to composition on the left with translations. $(2q-1)$ of these degrees of freedom will be used to "normalize" the Fatou coordinates. The remaining degree of freedom will be used later to adjust the families $\Psi^{0,\infty}_{j,\epsilon,\pm}$ so that they become 1-summable in ϵ.

Whenever possible, we will drop the lower indices $\pm$ referring to the sectors.

Lemma 3.2 *[13] The maps $\Psi^{0,\infty}_{j,\epsilon,\pm}(W) - W$ can be expanded as Fourier series with constant terms $A^{0,\infty}_{j,\epsilon,\pm}$. It is possible to compose the Fatou coordinates with translations so that all $A^{0,\infty}_{j,\epsilon,\pm} = A^{0,\infty}_{\epsilon}$ for*

$$A^0_\epsilon = -A^\infty_\epsilon = \pi i a/q.$$

Definition 3.3 A set of Fatou coordinates $\Phi^{\pm}_{j,\epsilon}$, $j = 1, \dots q$, such that the corresponding transition maps $\Psi^{0,\infty}_{j,\epsilon}$, $j = 1, \dots, q$, have constant terms as in Lemma 3.2, is called a *normalized set of Fatou coordinates.*

From now on, we will only consider normalized set of Fatou coordinates.

Proposition 3.4 *Here we drop the lower indices $\pm$ in the $\Psi^{0,\infty}_{j,\epsilon,\pm}$.*

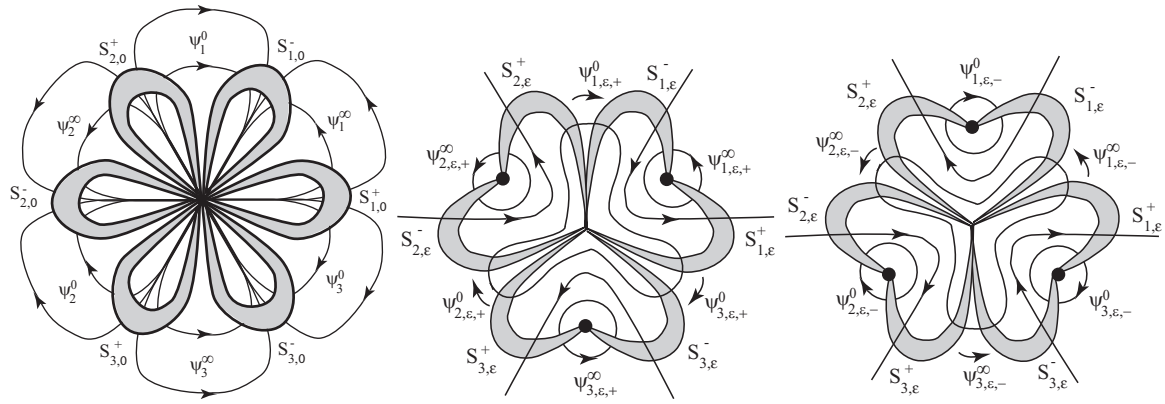
1. *Each map $\Psi^{0,\infty}_{j,\epsilon}$ commutes with the translation by 1: $\Psi^{0,\infty}_{j,\epsilon} \circ T_1 = T_1 \circ \Psi^{0,\infty}_{j,\epsilon}$. Hence $\Psi^{\infty}_{j,\epsilon}$ (resp. $\Psi^{0}_{j,\epsilon}$) induces a mapping $\hat{\Psi}^{\infty}_{j,\epsilon}$ (resp. $\hat{\Psi}^{0}_{j,\epsilon}$) defined on an open set of the cylinder $\mathbb{C}/\mathbb{Z}$ with values in $\mathbb{C}/\mathbb{Z}$.*

2. *Using the exponential function $W \mapsto w = E(W) = \exp(-2i\pi W)$, we can identify $\mathbb{C}/\mathbb{Z}$ with the sphere minus two points : $\mathbb{CP}^1 \setminus \{0, \infty\}$. The upper end of the cylinder $\mathbb{C}/\mathbb{Z}$, corresponds to $\infty \in \mathbb{CP}^1$ and the lower end to 0. Conjugating $\Psi^{0,\infty}_{j,\epsilon}$ with E yields analytic diffeomorphisms $\psi^{0,\infty}_{j,\epsilon} = E \circ \Psi^{0,\infty}_{j,\epsilon} \circ E^{-1}$. $\psi^{0}_{j,\epsilon}$ (resp. $\psi^{\infty}_{j,\epsilon}$) is defined in the neighborhood of 0 and (resp. ∞) on $\mathbb{CP}^1$ and such that $\psi^{0}_{j,\epsilon}(0) = 0$ (resp. $\psi^{\infty}_{j,\epsilon}(\infty) = \infty$).*

3. *The functions $\psi^{0,\infty}_{j,\epsilon,\pm}$ depend analytically on $\epsilon \neq 0$ in $V_{\delta,\pm}$ and are continuous in ϵ at $\epsilon = 0$.*

4. *The derivatives of $\psi^{0,\infty}_{j,\epsilon}$ are given by*

$$\begin{cases} (\psi^{0}_{j,\epsilon})'(0) = \exp(2\pi^2 a/q), \\ (\psi^{\infty}_{j,\epsilon})'(\infty) = \exp(2\pi^2 a/q). \end{cases} \tag{3.2}$$



(a) $\epsilon = 0$ (b) $\epsilon \in V_{\delta,+}$, $\arg \epsilon = \pi/2$ (c) $\epsilon \in V_{\delta,-}$, $\arg \epsilon = -\pi/2$

Figure 5: The maps $\psi_{j,\epsilon}$ for different values of ϵ

Since $g_\epsilon = f_\epsilon^n$, it happens that only $\Psi^{0,\infty}_{1,\epsilon}$ are independent, and the other $\Psi^{0,\infty}_{j,\epsilon}$, $j > 1$ are conjugate to them by translations.

Proposition 3.5 *We consider a map g_ϵ as in* (2.11)*, being the q-th iterate of a map f_ϵ as in* (2.8)*, the corresponding lifted diffeomorphism G_ϵ, and a normalized set of Fatou coordinates on either $V_{\delta,+}$ or $V_{\delta,-}$.*

1. *Let σ defined by $\sigma(j) = j + p \pmod q$ be the shift which represents the iterates of* $\exp(2\pi i/q)$ *under multiplication by* $\exp(2\pi i p/q)$*. Then*

$$\Psi^{0,\infty}_{\sigma(j),\epsilon} = T_{\frac{1}{q}} \circ \Psi^{0,\infty}_{j,\epsilon} \circ T_{-\frac{1}{q}}. \tag{3.3}$$

2. *Let $\tau = \exp\left(\frac{2\pi i}{q}\right)$. Then,*

$$\psi^{0,\infty}_{\sigma(j),\epsilon} = L_{\tau^{-1}} \circ \psi^{0,\infty}_{j,\epsilon} \circ L_\tau. \tag{3.4}$$

3. *Once $\Phi^{\pm}_{1,\epsilon}$ is chosen, the other Fatou coordinates can be taken such that*

$$\Phi^{\pm}_{\sigma(j),\epsilon} \circ F_\epsilon = T_{\frac{1}{q}} \circ \Phi^{\pm}_{j,\epsilon}. \tag{3.5}$$

Definition 3.6 Two germs of analytic families f_ϵ and $\overline{f}_{\overline{\epsilon}}$ of diffeomorphisms with a fixed point at the origin are *conjugate* if there exists a germ of analytic diffeomorphism $H(\epsilon, z) = (k(\epsilon), h(\epsilon, z))$ fibered over the parameter space such that

$$h_\epsilon \circ f_\epsilon = \overline{f}_{k(\epsilon)} \circ h_\epsilon, \tag{3.6}$$

where $h_\epsilon(z) = h(\epsilon, z)$.

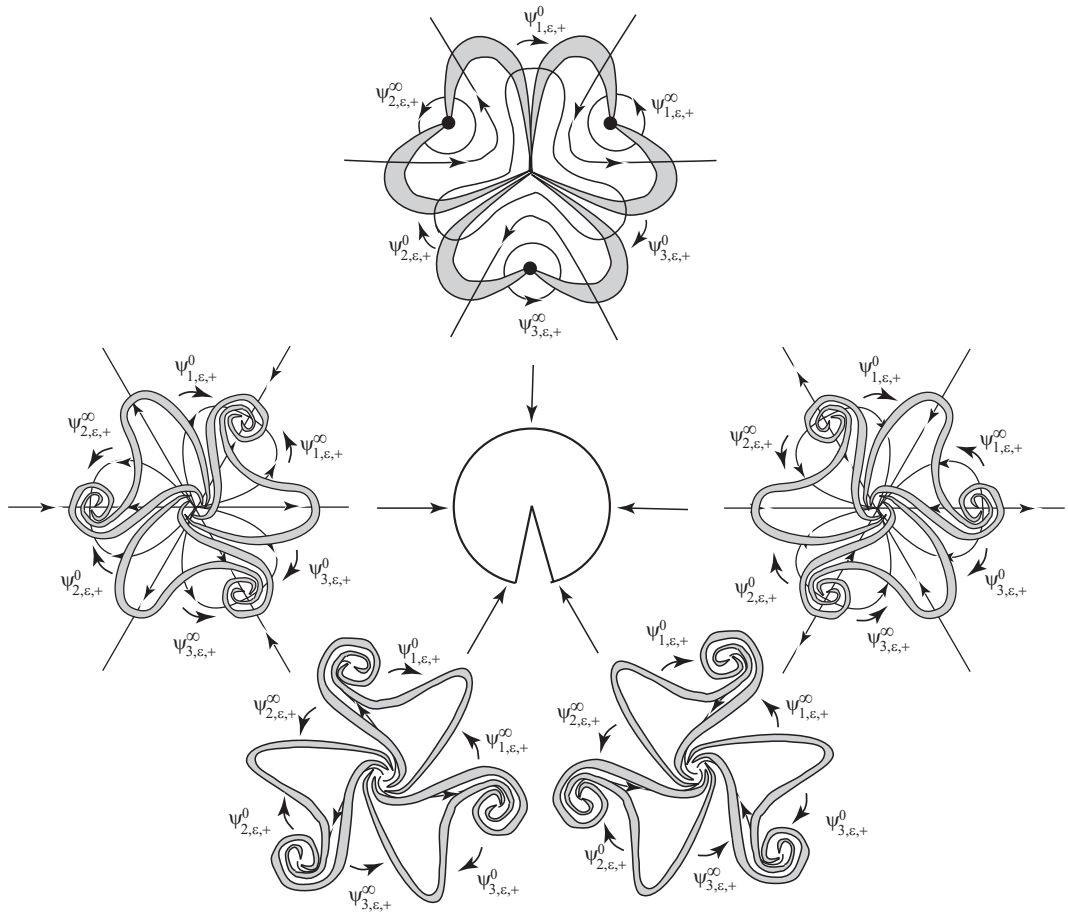


Figure 6: The crescents and maps $\psi^{0,\infty}_{j,\epsilon,+}$ for $\epsilon \in V_{\delta,+}$

Theorem 3.7 *We consider two germs of "prepared" families f_ϵ and $\overline{f}_\epsilon$ of the form* (2.8)*, i.e. so that the families of their q-th iterates g_ϵ and $\overline{g}_\epsilon$ are prepared of the form* (2.11)*. We choose common sectors $V_{\delta,\pm}$ on which the previous analysis applies. Then the two families are conjugate if and only if they have the same formal invariants $a(\epsilon)$ and there exist analytic functions $C_\pm(\epsilon) : V_{\delta,\pm} \to \mathbb{C}^*$ bounded and bounded away from zero and an integer $m \in \{0, \ldots, q-i\}$, such that for any $\epsilon \in V_{\delta,\pm}$*

$$\psi^{0,\infty}_{j,\epsilon,\pm} = L_{(C_\pm(\epsilon))^{-1}} \circ \overline{\psi}^{0,\infty}_{j+m,\epsilon,\pm} \circ L_{C_\pm(\epsilon)}. \tag{3.7}$$

Definition 3.8 The families $\left(a(\epsilon), \left(\psi^{0,\infty}_{j,\epsilon,\pm}\right) / \sim\right)$, where $\sim$ is defined in (3.7) is called the *modulus* of the germ of prepared family $\mathcal{F}$. The germ of analytic map $a(\epsilon)$ is called the *formal part* of the modulus, while $\left(\psi^{0,\infty}_{j,\epsilon,\pm}\right) / \sim$ is called the *analytic part.*

Remark 3.9 Note that with the notation we have chosen, the direction of the maps $\psi^{0,\infty}_{j,\epsilon,\pm}$ corresponds to identification of orbits when following the dynamics of g_ϵ forward. The maps

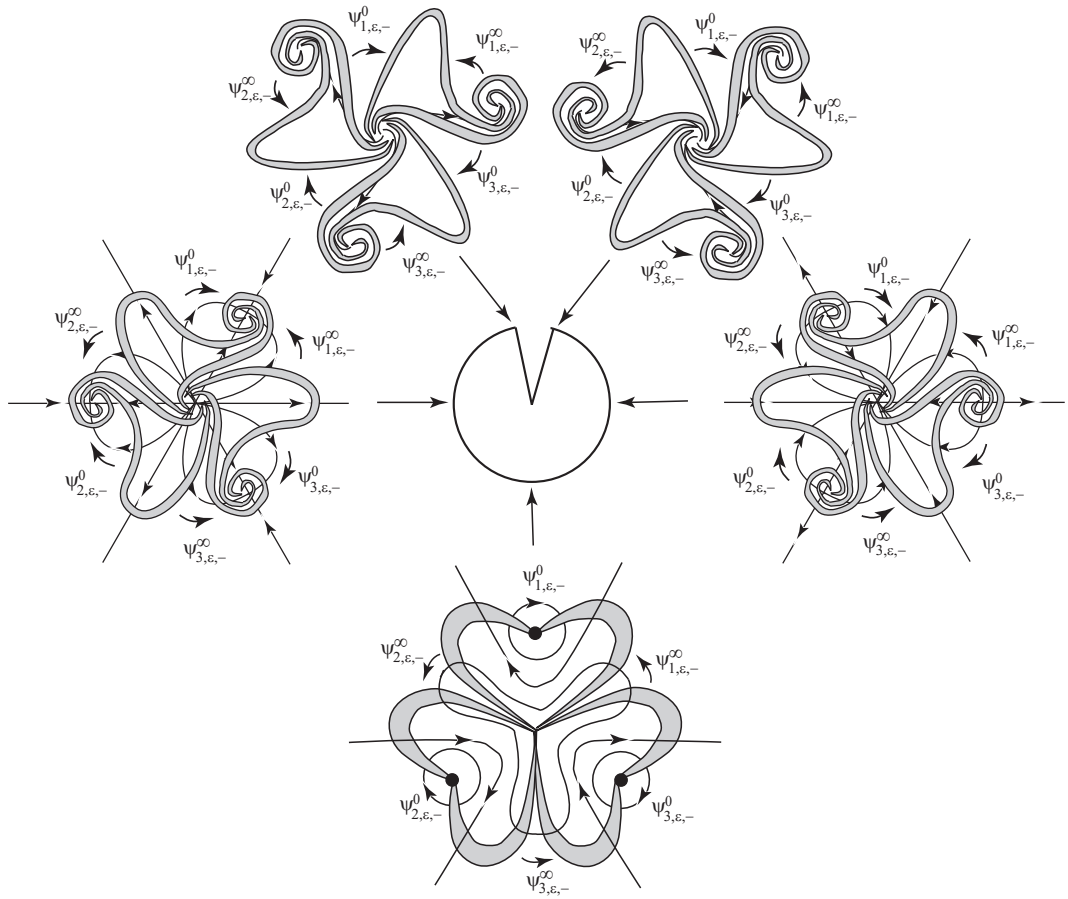


Figure 7: The crescents and maps $\psi^{0,\infty}_{j,\epsilon,-}$ for $\epsilon \in V_{\delta,-}$

$\psi^\infty_{j,\epsilon,\pm}$ (resp. $\psi^0_{j,\epsilon,\pm}$) are defined in the regions where the dynamics of g_ϵ near the boundary of U is in the positive (resp. negative) direction.

3.1 The Lavaurs phase

Proposition 3.10 *1. For $V_{\delta,+}$, the q Lavaurs translations are the maps*

$$T_{j,\epsilon,+} = \Phi^+_{j,\epsilon,+} \circ T_{-q\alpha(\epsilon)} \circ (\Phi^-_{j,\epsilon,+})^{-1} : Q^-_{j,+} \to Q^+_{j,+}. \tag{3.8}$$

2. For $V_{\delta,-}$, the q Lavaurs translations are the maps

$$T_{j,\epsilon,-} = \Phi^+_{j+1,\epsilon,-} \circ T_{-q\alpha(\epsilon)} \circ (\Phi^-_{j,\epsilon,-})^{-1} : Q^-_{j,-} \to Q^+_{j+1,-}. \tag{3.9}$$

3. When the Fatou coordinates are normalized, the Lavaurs translations are given by

$$T_{j,\epsilon,\pm}(W) = W \mp \left(\frac{2\pi i}{q\epsilon} + \frac{\pi i a}{q} \right). \tag{3.10}$$

3.2 The Glutsyuk point of view

It is also possible to take admissible lines parallel to the lines of holes as in Figure 8 when we limit ourselves to values of ϵ in $V_r \cup V_l$, which we call the *Glutsyuk domain*. Then the

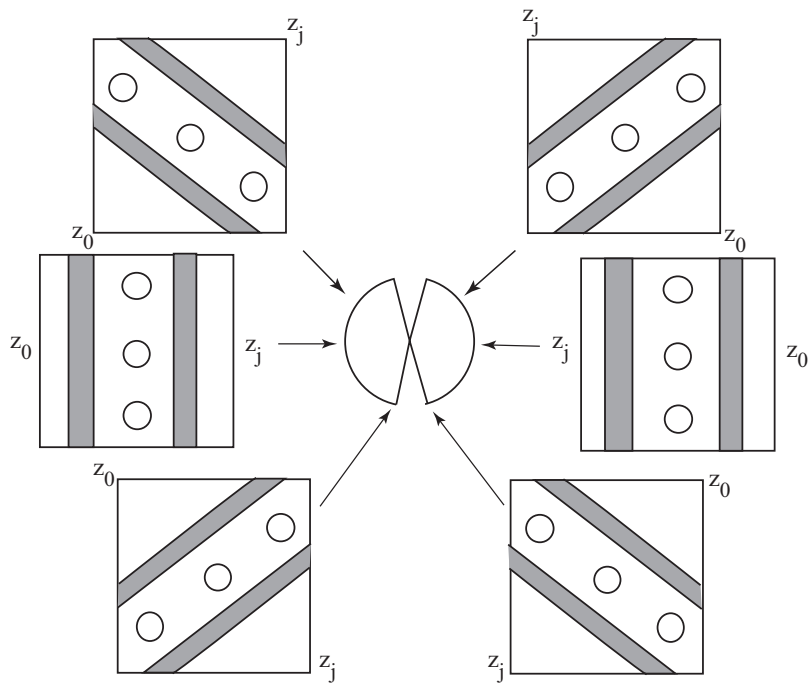


Figure 8: Continuous families of admissible lines and strips for ϵ in the Glutsyuk domain (for the sake of simplicity we have not drawn the ramification of S_ϵ at the holes)

fundamental domains are tori since G_ϵ commutes with $T_{q\alpha}$ (details as in [7]). The Fatou coordinates on the associated translation domains yield analytic changes of coordinates to the model family in the neighborhood of each of the fixed points of g_ϵ: these are named $\Phi^G_{j,\epsilon,r}$ and $\Phi^G_{j,\epsilon,l}$ for those covering a neighborhood of z_j and $\Phi^{G,j}_{0,\epsilon,r}$ and $\Phi^{G,j}_{0,\epsilon,l}$ for those covering a neighborhood of z_0 (there are q of these, one in each sheet of the covering). The lower index is r (resp. l) if $\epsilon \in V_r$ (resp. $\epsilon \in V_l$). As in the proof of Proposition 3.5, we can show that they can be chosen so as to satisfy (for $* \in \{r, l\}$)

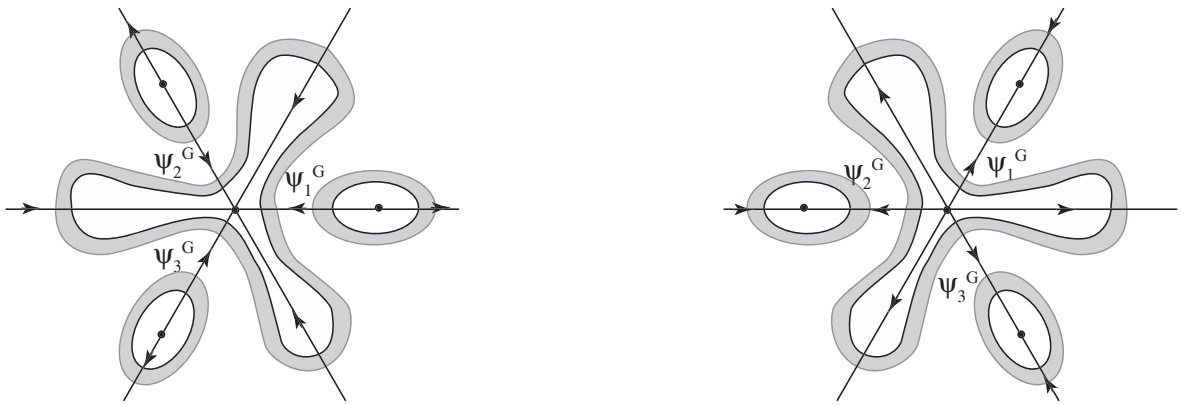
$$\begin{array}{rcl} \Phi^G_{\sigma(j),\epsilon,*}(F_\epsilon(Z)) & = & \Phi^G_{j,\epsilon,*}(Z) + \frac{1}{q}, \\ \Phi^{G,\sigma(j)}_{0,\epsilon,*}(F_\epsilon(Z)) & = & \Phi^{G,j}_{0,\epsilon,*}(Z) + \frac{1}{q}. \end{array} \tag{3.11}$$

From the shape of the Riemann surface as in Figure 2, it is clear that, for $* \in \{r, l\}$, the domain of any $\Phi^G_{j,\epsilon,*}$ intersects the domain of $\Phi^{G,j}_{0,\epsilon,*}$.

The transitions between the Fatou coordinates are given by (Figures 8 and 9)

$$\begin{array}{rcl} \Psi^G_{j,\epsilon,r} & = & \Phi^{G,j}_{0,\epsilon,r} \circ (\Phi^G_{j,\epsilon,r})^{-1}, \\ \Psi^G_{j,\epsilon,l} & = & \Phi^G_{j,\epsilon,l} \circ (\Phi^{G,j}_{0,\epsilon,l})^{-1}. \end{array} \tag{3.12}$$

They depend continuously on ϵ as $\epsilon \to 0$. At the limit the domain becomes disconnected and the $\Psi^G_{j,\epsilon,\pm}$ tend to Ψ^0_j on one half of the domain and Ψ^∞_j on the other half.



(a) $\epsilon \in V_l$ (b) $\epsilon \in V_r$

Figure 9: The Glutsyuk maps

Proposition 3.11 *For $* \in \{L, R\}$,*

$$\Psi^G_{\sigma(j),\epsilon,*} = T_{\frac{1}{q}} \circ \Psi^G_{j,\epsilon,*} \circ T_{-\frac{1}{q}}. \tag{3.13}$$

Remark 3.12 The projection by p_ϵ of a Glutsyuk translation domain on which we can bring the family to the model yields a neighborhood of one fixed point of f_ϵ on which the family is conjugate to the model.

3.3 The Martinet-Ramis point of view

In [8], Martinet and Ramis present the orbit space of f_0 as the union of two spheres identified in the neighborhoods of 0 and ∞ by two germs of diffeomorphisms (instead of our descriptions with $2q$-spheres and $2q$ germs of diffeomorphisms). Their description carries over to the unfolding.

We consider a normalized set of (Lavaurs) Fatou coordinates generated by admissible lines $\ell^\pm_j(\epsilon)$. These lines together with their images $G_\epsilon(\ell^\pm_j(\epsilon))$ determine strips $\hat{C}^\pm_{j,\epsilon}$. Their images by p_ϵ are crescents $C^\pm_{j,\epsilon}$. Their quotient under g_ϵ are conformally equivalent to $\mathbb{CP}^1 \setminus \{0, \infty\}$ by Proposition 3.4. We call these quotient spaces $S^\pm_{j,\epsilon}$.

Proposition 3.13 *[13] Over each sector $V_{\delta,\pm}$, the orbit space of f_ϵ is described by the union of the two spheres $S^+_{1,\epsilon} \cup S^-_{1,\epsilon}$ identified in the neighborhood of ∞ (resp. 0) by $\boldsymbol{\psi}^\infty_\epsilon$ (resp. $\boldsymbol{\psi}^0_\epsilon$) where*

$$\boldsymbol{\psi}^{0,\infty}_\epsilon : S^+_{1,\epsilon} \to S^-_{1,\epsilon},$$

are defined by

$$\begin{cases} \boldsymbol{\psi}^\infty_\epsilon = \psi^\infty_{1,\epsilon} \\ \boldsymbol{\psi}^0_\epsilon = \psi^0_{1,\epsilon} \circ L_{\tau^m} \end{cases} \tag{3.14}$$

with

$$L_{\tau^m}(w) = \exp\left(\frac{2\pi i m}{q}\right) w, \qquad \textit{where} \qquad mp \equiv -1 \ (mod\ q). \tag{3.15}$$

4 The compatibility condition

We decide to work in the Martinet-Ramis point of view described in Section 3.3.

4.1 The renormalized maps

Using normalized Fatou coordinates and conjugating the Lavaurs translations by E yields Lavaurs linear maps, $L_{\nu_\pm(\epsilon),\pm}$, where

$$\nu_\pm(\epsilon) = \exp\left(\mp\left(\frac{4\pi^2}{q\epsilon} + \frac{2\pi^2 a}{q}\right)\right). \tag{4.1}$$

We simply note them $L_{\nu_\pm}$.

Proposition 4.1 *1. For $\epsilon \in V_{\delta,+} \setminus \{0\}$, there exist for the map g_ϵ:*

i) renormalized return maps: $k^\pm_{1,\epsilon,+} : S^\pm_{1,\epsilon} \to S^\pm_{1,\epsilon}$, defined by

$$k^+_{1,\epsilon,+} = L_{\nu_+} \circ \boldsymbol{\psi}^\infty_{\epsilon,+}, \qquad k^-_{1,\epsilon,+} = \boldsymbol{\psi}^\infty_{\epsilon,+} \circ L_{\nu_+}; \tag{4.2}$$

ii) renormalized return maps: $k^\pm_{0,\epsilon,+} : S^\pm_{1,\epsilon} \to S^\pm_{1,\epsilon}$ defined by

$$k^+_{0,\epsilon,+} = L_{\nu_+} \circ \boldsymbol{\psi}^0_{\epsilon,+}. \qquad k^-_{0,\epsilon,+} = \boldsymbol{\psi}^0_{\epsilon,+} \circ L_{\nu_+}. \tag{4.3}$$

2. For $\epsilon \in V_{\delta,-} \setminus \{0\}$, there exist for the map g_ϵ:

i) renormalized return maps: $k^\pm_{1,\epsilon,-} : S^\pm_{1,\epsilon} \to S^\pm_{1,\epsilon}$ defined by

$$k^+_{1,\epsilon,-} = L_{\nu_-} \circ L_{\tau^{-m}} \circ \boldsymbol{\psi}^0_{\epsilon,-}, \qquad k^-_{1,\epsilon,-} = \boldsymbol{\psi}^0_{\epsilon,-} \circ L^{-1}_{\tau^{-m}} \circ L_{\nu_-}; \tag{4.4}$$

ii) renormalized return maps: $k^\pm_{0,\epsilon,-} : S^\pm_{1,\epsilon} \to S^\pm_{1,\epsilon}$ defined by

$$k^+_{0,\epsilon,-} = L_{\nu_-} \circ L_{\tau^{-m}} \circ \boldsymbol{\psi}^\infty_{\epsilon,-}, \qquad k^-_{0,\epsilon,-} = \boldsymbol{\psi}^\infty_{\epsilon,-} \circ L_{\tau^{-m}} \circ L_{\nu_-}. \tag{4.5}$$

Proof. We calculate one case. The others are done in a similar way. To decide which maps to compose, it is best to use Figure 5. The basic ingredients are the following:

$$\psi^{0,\infty}_{j-1} = L_{\tau^{-m}} \circ \psi^{0,\infty}_j \circ L_{\tau^m}, \tag{4.6}$$

$$L_{\tau^m} : S^\pm_{j-1} \to S^\pm_j. \tag{4.7}$$

1. ii) We have $\psi^0_{q-1,\epsilon,+} : S^+_1 \to S^-_{q-1}$. Hence, for $k^+_{0,\epsilon,+}$, we compose it on the left with L_{ν_+}, thus having a map $\psi^0_{q-1,\epsilon,+} \circ L_{\nu_+} : S^+_1 \to S^+_{q-1}$.We finally compose on the left with L_{τ^m} to get $L_{\tau^m} \circ L_{\nu_+} \circ \psi^0_{q-1,\epsilon,+} : S^+_1 \to S^+_1$ (using (4.7)).

$$\begin{aligned} k^+_{0,\epsilon,+} &= L_{\tau^m} \circ L_{\nu_+} \circ \psi^0_{q-1,\epsilon,+} \\ &= L_{\nu_+} \circ \psi^0_{1,\epsilon,+} \circ L_{\tau^m} = L_{\nu_+} \circ \boldsymbol{\psi}^0_{\epsilon,+}. \end{aligned} \tag{4.8}$$

We finally use (4.6) and (3.14) to get the result.

4.2 The compatibility condition

The two sectors $V_{\delta,\pm}$ intersect in two smaller sectors V_l and V_r (Figure 1). Over these sectors we have two different moduli representing the same family. The compatibility condition expresses that these two moduli encode the same dynamics. Note that, on V_l and V_r, all maps $k^{\pm}_{j,\epsilon,\pm}$, $j \in \{0,1\}$, of Proposition 4.1 are linearizable.

It is possible to recover the Glutsyuk modulus (3.12) (or its conjugate by E) from the renormalized return maps $k_{j,\epsilon,\pm}$ of Proposition 4.1 in w-coordinate (or either W-coordinate as in [1]). This is rather straightforward and could be written in all details as in [1]. We have chosen to use the Martinet-Ramis point of view in order to avoid working on the q-sheeted space where the details look messier than they really are.

Later in Section 5.1, we will give a geometric justification of this trick, when realizing each germ of family of diffeomorphisms as a germ of a family of holonomies of a separatrix of a family of saddles.

Theorem 4.2 *We consider a germ of family of diffeomorphisms with modulus defined as before. For $\epsilon \in V_l \cup V_r$, let $h^{\pm}_{j,\epsilon,\pm}$, $j \in \{0,1\}$, the map tangent to the identity linearizing $k^{\pm}_{j,\epsilon,\pm}$. There exists constant $D(l,\epsilon)$, $D(r,\epsilon)$, $D'(l,\epsilon)$ and $D'(r,\epsilon)$ such that*

$$h^{+}_{1,\epsilon,+} \circ (h^{+}_{0,\epsilon,+})^{-1} = L_{D(l,\epsilon)} \circ h^{-}_{1,\epsilon,-} \circ (h^{-}_{0,\epsilon,-})^{-1} \circ L_{D'(l,\epsilon)}, \qquad \epsilon \in V_l, \tag{4.9}$$

$$h^{+}_{0,\epsilon,+} \circ (h^{+}_{1,\epsilon,+})^{-1} = L_{D(r,\epsilon)} \circ h^{-}_{0,\epsilon,-} \circ (h^{-}_{1,\epsilon,-})^{-1} \circ L_{D'(r,\epsilon)}, \qquad \epsilon \in V_r. \tag{4.10}$$

Proof. The proof is similar to that of [1] made in the coordinate $W = E^{-1}(w)$, but here we write it directly in the w-coordinate. Because the fixed points of g_ϵ are linearizable, a fundamental domain is given by a torus (see Figure 9) of modulus $\frac{2\pi i}{\mu_j}$. We look for a covering map on a sphere minus two points. This sphere is identified to an infinite cylinder, infinitely winding over the torus. If we choose to normalize the diameter of the cylinder to 1, then points w and $\frac{2\pi i}{\mu_j} w$ are sent to the same torus point. So looking for this spherical coordinate near a fixed point is equivalent to linearizing the renormalized return map. This linearizing map is unique up to composition on the left with linear maps. Hence, the comparison of these linearizing maps, an expression of the Glutsyuk modulus in these spherical coordinates, is unique up to composition on the left and on the right with linear maps. In (4.9), a first expression of the Gutsyuk modulus is given by $h^{+}_{1,\epsilon,+} \circ (h^{+}_{0,\epsilon,+})^{-1}$ using the description of the dynamics on V_+. A second expression is given by $h^{-}_{1,\epsilon,-} \circ (h^{-}_{0,\epsilon,-})^{-1}$ using the description of the dynamics on V_-. Thus, the two must coincide up to composition with linear diffeomorphisms. The same reasoning is done on V_r. □

Remark 4.3 On V_l or V_r, the choice of $h^{\pm}_{j,\epsilon,+}$ or $h^{\pm}_{j,\epsilon,-}$ comes from the fact that we must compare in the same region of z-space. Indeed, the constants $\nu_{\pm}(\epsilon)$ defined in (4.1) are exponentially small or large.

Theorem 4.4 *It is possible to choose Fatou coordinates such that the constant $D(l,\epsilon) \equiv (\boldsymbol{\psi}^{\infty}_{\epsilon,-})'(\infty)$ (resp. $D(r,\epsilon) \equiv (\boldsymbol{\psi}^{0}_{\epsilon,-})'(0)$). Under this condition, there exists $A > 0$ such that, for $\epsilon \in V_l \cup V_r$:*

$$\left| \boldsymbol{\psi}^{0,\infty}_{\epsilon,-} - \boldsymbol{\psi}^{0,\infty}_{\epsilon,+} \right| = O\left(\exp\left(-\frac{A}{|\epsilon|} \right) \right). \tag{4.11}$$

Proof. It follows the ideas of [13]. The equation (4.9) allows to compute the involved $h^{\pm}_{j,\epsilon,\pm}$:

$$\begin{cases} h^{+}_{0,\epsilon,+} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^{-}_{1,\epsilon,-} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ (h^{-}_{0,\epsilon,-})^{-1} = \boldsymbol{\psi}^{\infty}_{\epsilon,-} \circ L_{B_1(\epsilon)} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^{+}_{1,\epsilon,+} = L_{B_1(\epsilon)} \circ \boldsymbol{\psi}^{\infty}_{\epsilon,+} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \end{cases} \tag{4.12}$$

where

$$B_1(\epsilon) = (\boldsymbol{\psi}^{\infty}_{\epsilon,-})'(\infty) = (\boldsymbol{\psi}^{\infty}_{\epsilon,+})'(\infty). \tag{4.13}$$

There is one degree of freedom for each of the four Fatou coordinates $\Phi^{\pm}_{j,\epsilon,\pm}$. Two are used for normalizing the Fatou coordinates. One is used for adjusting $D(l,\epsilon)$ and the other for adjusting $D(r,\epsilon)$. The details are written below. Suppose now that we have adjusted the Fatou coordinates so as to get $D(l,\epsilon) \equiv (\boldsymbol{\psi}^{\infty}_{\epsilon,-})'(\infty)$.

This implies that

$$D'(\epsilon,l) \equiv ((\boldsymbol{\psi}^{\infty}_{\epsilon,-})'(\infty))^{-1} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right).$$

This, together with (4.12), allows to conclude that

$$\left|\boldsymbol{\psi}^{\infty}_{\epsilon,-} - \boldsymbol{\psi}^{\infty}_{\epsilon,+}\right| = O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right). \tag{4.14}$$

To get the other part, namely

$$\left|\boldsymbol{\psi}^{0}_{\epsilon,-} - \boldsymbol{\psi}^{0}_{\epsilon,+}\right| = O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \tag{4.15}$$

we remark that we can obtain the $h^{\mp}_{j,\epsilon,\pm}$ from the $h^{\pm}_{j,\epsilon,\pm}$ and then replace in (4.9). Indeed,

$$\begin{cases} h^{-}_{j,\epsilon,+} = L_{\nu_+^{-1}} \circ h^{+}_{j,\epsilon,+} \circ L_{\nu_+}, \qquad j \in \{0,1\}, \\ h^{-}_{j,\epsilon,-} = L_{\nu_-^{-1}\tau^m} \circ h^{+}_{j,\epsilon,-} \circ L_{\nu_-\tau^{-m}}, \qquad j \in \{0,1\}, \\ \nu_+ = \nu_-^{-1}, \\ (\psi^0_\epsilon)'(0) = \tau^m (\psi^\infty_\epsilon)'(\infty). \end{cases} \tag{4.16}$$

The substitution in (4.9) yields

$$h^{-}_{1,\epsilon,+} \circ (h^{-}_{0,\epsilon,+})^{-1} = L_{D(l,\epsilon)\tau^m} \circ h^{+}_{1,\epsilon,-} \circ (h^{+}_{0,\epsilon,-})^{-1} \circ L_{D'(l,\epsilon)\tau^{-m}}. \tag{4.17}$$

Now,

$$\begin{cases} (h^{-}_{0,\epsilon,+})^{-1} = \psi^{0}_{\epsilon,+} \circ L_{B_2(\epsilon)} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^{+}_{1,\epsilon,-} = L_{B_2(\epsilon)} \circ \psi^{0}_{\epsilon,-} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ (h^{+}_{0,\epsilon,-})^{-1} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^{-}_{1,\epsilon,+} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \end{cases} \tag{4.18}$$

with

$$B_2(\epsilon) = ((\psi^0_{\epsilon,+})'(0))^{-1}. \tag{4.19}$$

Replacing in (4.17) yields (4.15).

The right side is done similarly, using (4.10). The calculations yield

$$\begin{cases} h^+_{1,\epsilon,+} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^-_{0,\epsilon,-} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ (h^-_{1,\epsilon,-})^{-1} = \boldsymbol{\psi}^0_{\epsilon,-} \circ L_{B_3(\epsilon)} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^+_{0,\epsilon,+} = L_{B_3(\epsilon)} \circ \boldsymbol{\psi}^0_{\epsilon,+} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \end{cases} \tag{4.20}$$

with

$$B_3(\epsilon) = ((\boldsymbol{\psi}^0_{\epsilon,-})'(0))^{-1}. \tag{4.21}$$

We use the last degree of freedom in the choice of Fatou coordinates (see details below) to set

$$D(r,\epsilon) \equiv ((\boldsymbol{\psi}^0_{\epsilon,-})'(0))^{-1},$$

yielding

$$D'(r,\epsilon) \equiv (\boldsymbol{\psi}^0_{\epsilon,-})'(0) + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right).$$

We also have

$$\begin{cases} h^-_{0,\epsilon,+} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^+_{1,\epsilon,-} = id + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ h^+_{0,\epsilon,-} = L_{B_4(\epsilon)} \circ \boldsymbol{\psi}^\infty_{\epsilon,-} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \\ (h^-_{1,\epsilon,+})^{-1} = \boldsymbol{\psi}^\infty_{\epsilon,+} \circ L_{B_4(\epsilon)} + O\left(\exp\left(-\frac{A}{|\epsilon|}\right)\right), \end{cases} \tag{4.22}$$

with

$$B_4(\epsilon) = (\boldsymbol{\psi}^\infty_{\epsilon,-})'(\infty). \tag{4.23}$$

We are now left to explain how we adjust the Fatou coordinates, so as to get the special values of the constants $D(l,\epsilon)$ and $D(r,\epsilon)$. We look for functions $D_\pm(\epsilon)$ defined respectively on $V_\pm$ such that

$$D_+(\epsilon)(D_-(\epsilon))^{-1} = \begin{cases} B_1(\epsilon)(D(l,\epsilon))^{-1}, & \epsilon \in V_l, \\ B_3(\epsilon)(D(r,\epsilon))^{-1}, & \epsilon \in V_r. \end{cases}$$

Indeed, we have (4.9). The degree of freedom allows to change $\boldsymbol{\psi}^{0,\infty}_{\epsilon,\pm}$ to $L_{D_\pm(\epsilon)} \circ \boldsymbol{\psi}^{0,\infty}_{\epsilon,\pm} \circ L_{(D_\pm(\epsilon))^{-1}}$. Then, this changes the functions $h^\pm_{j,\epsilon,\pm}$ to $L_{D_\pm(\epsilon)} \circ h^\pm_{j,\epsilon,\pm} \circ L_{(D_\pm(\epsilon))^{-1}}$. The new constant $D_{new}(l,\epsilon)$ (resp. $D_{new}(r,\epsilon)$) in the compatibility condition becomes

$$\begin{cases} D_{new}(l,\epsilon) = D(l,\epsilon)D_+(\epsilon)(D_-(\epsilon))^{-1}, & \epsilon \in V_l, \\ D_{new}(r,\epsilon) = D(r,\epsilon)D_+(\epsilon)(D_-(\epsilon))^{-1}, & \epsilon \in V_r. \end{cases}$$

The functions $D_\pm(\epsilon)$ are just found as solutions of the second Cousin problem. □

5 Link between the holonomies of the two separatrices of a saddle

We start by justifying why it is relevant to study the link between the holonomies of the two separatrices of a saddle.

5.1 The geometric justification of the reduction to the case $q = 1$

Below, we will use the following trick to reduce to the case $q = 1$. Indeed, it is shown in [13] that, for general multiplier $\exp\left(2\pi i \frac{p}{q}\right)$, the modulus is given by

(i) the codimension 1,

(ii) the multiplier $\exp\left(2\pi i \frac{p}{q}\right)$,

(iii) the canonical parameter,

(iv) the unfolding $a(\epsilon)$ of the formal invariant,

(v) the family of pairs $(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})_{\epsilon \in V_\pm}$.

Only (ii) depends on the multiplier. Moreover, (v) depends only of $a \pmod{\frac{1}{2\pi i}}$.

We will show below in Section 6 that, given any $p_1, q_1 \in \mathbb{N}^*$ such that $(p_1, q_1) = 1$ and $p_1 \leq q_1$, and any $p_1' \in \mathbb{N}^*$ such that $p_1' \equiv -p_1 \pmod{q_1}$, for any unfolding f_ϵ of a germ of resonant diffeormorphism with multiplier (for $\epsilon = 0$) of the form $\exp\left(2\pi i \frac{p_1'}{q_1}\right)$ there exists an unfolding of germ of vector field X_ϵ at a resonant saddle point with hyperbolicity ratio $\frac{p_1'}{q_1}$ for $\epsilon = 0$, such that f_ϵ is the holonomy of the x-separatrix of X_ϵ. In practice, we can suppose $p_1' \leq q_1$ (since otherwise we can multiply the vector field by $\frac{q_1}{p_1'}$ to attain this case). We apply this construction to find an unfolding of a vector field with eigenvalues 1 and $-\frac{p_1'}{q_1}$ where

$$p_1' = \begin{cases} q_1 - p_1, & p_1 < q_1, \\ 1, & q_1 = 1. \end{cases}$$

The holonomy h_x of the x-separatrix has multiplier $\exp\left(2\pi i \frac{p_1}{q_1}\right)$, while the holonomy h_y of the y-separatrix has multiplier $\exp\left(2\pi i \frac{p_2}{q_2}\right)$, with

$$\exp\left(2\pi i \frac{p_2}{q_2}\right) = \exp\left(2\pi i \frac{q_1}{p_1}\right) = \exp\left(2\pi i \frac{q_1 - p_1}{p_1}\right)$$

for $\epsilon = 0$. We will derive the relation between the canonical parameter $\epsilon_1 = \epsilon$ of $h_x = h_1$ and the canonical parameter ϵ_2 of $h_y = h_2$. We also compute the relation between their formal invariants. Moreover, we will show that for the analytic part of the modulus $\boldsymbol{\psi}^{0,\infty}_{\epsilon_1,x}$ of h_x and $\boldsymbol{\psi}^{0,\infty}_{\epsilon_2,y}$ of h_y, we have the relation:

$$\boldsymbol{\psi}^{0,\infty}_{\epsilon_1,x} = H \circ (\boldsymbol{\psi}^{\infty,0}_{\epsilon_2,y})^{-1} \circ H,$$

where

$$H(w) = \frac{1}{w}.$$

So our construction has produced a new map h_y with a multiplier $\exp\left(2\pi i \frac{p_2}{q_2}\right)$ with $q_2 < q_1$.

We can iterate the construction until $q_n = 1$. Now, we can proceed backwards. We start with a family h_n which is realizable. Then we realize h_{n-1}, etc., until we realize h_1. Hence, it suffices to derive the sufficient condition for realizability for a family of resonant diffeomorphisms in the case of a multiplier equal to 1.

Before deriving this sufficient condition, we first write the details of the correspondence between the moduli of the two separatrices of a saddle.

5.2 Link between the holonomies of the two separatrices of a saddle

If we consider a germ of generic family of analytic vector fields unfolding a codimension 1 resonant saddle with hyperbolicity ratio $\frac{p'}{q}$, it is possible, by an analytic change of coordinates and scaling of time, to bring it to a prepared form X_ϵ given by

$$X_\epsilon = \begin{cases} \dot{x} &= x \\ \dot{y} &= y[-(\frac{p'}{q} + \eta) + \alpha(\eta)u + O(u^2)], \end{cases} \tag{5.1}$$

with $u = x^{p'} y^q$, such that the holonomy of the x-separatrix on the section $x = 1$ is in prepared form (modulo a rotation in y of angle $\frac{\pi}{2q}$) with canonical parameter

$$\epsilon = 2\pi i p' \eta, \tag{5.2}$$

and the invariant manifold has the equation $u = \epsilon$.

If one looks at the holonomy of one separatrix for a germ of resonant saddle point, for instance the holonomy of the x-separatrix, it is possible to scale the x-variable so that the section $x = 1$ belongs to the domain of definition, U, of a representative of the germ and to scale y and the parameter so that the holonomy of the x-separatrix be prepared. We then have used all our degrees of freedom in scaling and it is not possible to simultaneously scale x and y so that $y = 1$ is included in U and the holonomy of the y-axis be prepared. We will prefer a different scaling: we rather choose to scale x and y so that $x = 1$ and $y = 1$ both are included in U. Using a change of variable tangent to the identity and a change of parameter

$$\eta = D\epsilon$$

we can suppose that the invariant manifold be given by

$$x^{p'} q^q = \epsilon.$$

Then, on $x = 1$ (resp. $y = 1$) the periodic points have equation $y^q = \epsilon$ (resp. $x^{p'} = \epsilon$). Let f_x (resp. f_y) be the holonomies of the x (resp. y) separatrix. Then the canonical parameters ϵ_x and ϵ_y for f_x (resp. f_y) are defined through

$$\begin{cases} (f_x^q)'(0) = \exp(-2\pi i p' \eta) = \exp(-2\pi i p' D\epsilon) = \exp(-\epsilon_x), \\ (f_y^{p'})'(0) = \exp(2\pi i q \frac{\eta}{1+\eta}) = \exp(2\pi i q \frac{D\epsilon}{1+D\epsilon}) = \exp(-\epsilon_y). \end{cases} \tag{5.3}$$

The formal parameters a_x (resp. a_y) of f_x (resp. f_y) are defined through

$$\begin{cases} (f_x^q)'(\epsilon_x) = \exp\left(\frac{q\epsilon_x}{1+a_x\epsilon_x}\right), \\ (f_y^{p'})'(\epsilon_y) = \exp\left(\frac{p'\epsilon_y}{1+a_y\epsilon_y}\right), \end{cases} \tag{5.4}$$

the latter definition yielding a well defined limit as $\epsilon_x \to 0$ (resp. $\epsilon_y \to 0$).

Proposition 5.1 *The canonical parameters ϵ_x and ϵ_y associated to the respective holonomies f_x and f_y of the x- and y- separatrices satisfy*

$$\epsilon_x = -\frac{2\pi i p'\epsilon_y}{\epsilon_y + 2\pi i q}. \tag{5.5}$$

In particular, the images of sectors $V_{\pm,x}(\delta)$ contain sectors $V_{\pm,y}(\delta')$ for some δ' and conversely. To prepare the family which is the unfolding of f_x (resp. f_y), an additional scaling $x \mapsto \tilde{x}$ (resp. $y \mapsto \tilde{y}$) is needed so that the equation of fixed points of f_x^q (resp $f_y^{p'}$) becomes $\tilde{x}^q = \epsilon_x$ (resp. $\tilde{y}^{p'} = \epsilon_y$). Moreover,

$$\epsilon_x \in i\mathbb{R} \Longleftrightarrow \epsilon_y \in i\mathbb{R}.$$

Proof. To study the holonomy f_x of the x-axis we consider the family in prepared form with canonical parameter ϵ_x for f_x. The family in prepared form is formally equivalent to a family

$$\begin{aligned} \dot{x} &= x(1 + A_x(\epsilon_x)u), \\ \dot{y} &= -\tfrac{p'}{q}y(1+\eta_x)(1 + B_x(\epsilon_x)u), \end{aligned} \tag{5.6}$$

where $\epsilon_x = 2\pi i p'\eta_x$, in which $u = \epsilon_x$ is an invariant manifold, yielding

$$B_x(\epsilon_x) = \frac{A_x(\epsilon_x)}{1+\eta_x} - \frac{1}{2\pi i p'(1+\eta_x)}. \tag{5.7}$$

Then the formal parameter $a(\epsilon_x)$ for f_x is simply $a_x(\epsilon_x) = A_x(\epsilon)$. □

Remark 5.2 Even if a scaling in the variable is missing so that a family be in prepared form we can still suppose that we have the same Fatou coordinates. We only need to compose the map p_ϵ^{-1} with the given scaling. We call the composition p_x^{-1} (resp. p_y^{-1}) and drop the index ϵ.

Theorem 5.3 *We consider a family* (5.1) *of vector fields in prepared form. Then*

(1) If ϵ_x (resp. ϵ_y) is the canonical parameter of the holonomy f_x of the x-axis (resp. the holonomy f_y of the y-axis) and $a_x(\epsilon_x)$ (resp. $a_y(\epsilon_y)$) is the formal parameter of f_x (resp. f_y) then

$$a_y(\epsilon_y) = A_y(\epsilon_y) = -\frac{p'}{q}a_x(\epsilon_x) + \frac{1}{2\pi i q}. \tag{5.8}$$

(2) There exists appropriate representatives $(\boldsymbol{\psi}^0_{x,\epsilon_x}, \boldsymbol{\psi}^\infty_{x,\epsilon_x})$ (resp. $(\boldsymbol{\psi}^0_{y,\epsilon_y}, \boldsymbol{\psi}^\infty_{y,\epsilon_y})$) of the modulus of f_x (resp. f_y) such that

$$\begin{cases} \boldsymbol{\psi}^0_{x,\epsilon_x} = H \circ (\boldsymbol{\psi}^\infty_{y,\epsilon_y})^{-1} \circ H^{-1} \\ \boldsymbol{\psi}^\infty_{x,\epsilon_x} = H \circ (\boldsymbol{\psi}^0_{y,\epsilon_y})^{-1} \circ H^{-1}, \end{cases} \tag{5.9}$$

where $H(w) = 1/w$.

(3) In particular, f_x and f_y^{-1} have the same analytic part of the modulus (but not the same formal invariant!): if $(\hat{\boldsymbol{\psi}}^0_{y,\epsilon_y}, \hat{\boldsymbol{\psi}}^\infty_{y,\epsilon_y}))$ is the analytic part of the modulus of f_y^{-1}, then

$$(\boldsymbol{\psi}^0_{x,\epsilon_x}, \boldsymbol{\psi}^\infty_{x,\epsilon_x}) = (\hat{\boldsymbol{\psi}}^0_{y,\epsilon_y}, \hat{\boldsymbol{\psi}}^\infty_{y,\epsilon_y}). \tag{5.10}$$

Proof.

(1) In order that the invariant manifold $u = \epsilon_x$ for (5.6) becomes $u' = \epsilon_y$ when we prepare (5.6) with x and y interchanged, we need to scale $(x, y) \mapsto (x', y') = (\alpha x, \beta y)$ with $\alpha^{p'}\beta^q = C = -\frac{q}{p'(1+\eta_x)} = -\frac{q(1+\eta_y)}{p'}$. This yields $B_y(\epsilon_y) = \frac{A_x(\epsilon_x)}{C}$ and $A_y(\epsilon_y) = \frac{B_x(\epsilon_x)}{C}$. From (5.7) we also get

$$B_y(\epsilon_y) = \frac{A_y(\epsilon_y)}{1+\eta_y} - \frac{1}{2\pi i q(1+\eta_y)} \tag{5.11}$$

as expected. The formal invariant $a_y(\epsilon_y) = A_y(\epsilon_y)$ for the holonomy f_y is then given by (5.8).

(2) We now consider the Dulac map Δ from a section $\{y = 1\}$ to a section $\{x = 1\}$ (see for instance [6]). The Dulac map is defined as follows: let $(x, 1)$ be a point of the section. We define $\Delta(x)$ as the endpoint of the lifting inside the leaf starting from $(x, 1)$ of the path from x to $\frac{x}{|x|}$ followed by the path from $\frac{x}{|x|}$ to 1 along the circle of radius 1 in the positive direction. Since both the map Δ and its inverse are ramified, they must be seen from the universal unfolding of a neighborhood of the origin in x-plane to the universal unfolding of a neighborhood of the origin in y-plane. We call f_x (resp. f_y) the holonomy of the x-axis (resp. y-axis). The map Δ satisfies

$$\begin{cases} \Delta \circ R_x = f_x^{-1} \circ \Delta, \\ \Delta^{-1} \circ R_y = f_y^{-1} \circ \Delta^{-1}, \end{cases} \tag{5.12}$$

where R_x (resp. R_y) is defined on the universal covering of a neighborhood of the origin in x-space (resp. y-space) by $R_x(x) = e^{2\pi i}x$ (resp. $R_y(y) = e^{2\pi i}y$). By choosing an adequate determination for Δ we have that $\Delta \circ \Delta^{-1} = id = \Delta^{-1} \circ \Delta$, from which it follows that

$$(R_y)^{-1} \circ \Delta = \Delta \circ f_y. \tag{5.13}$$

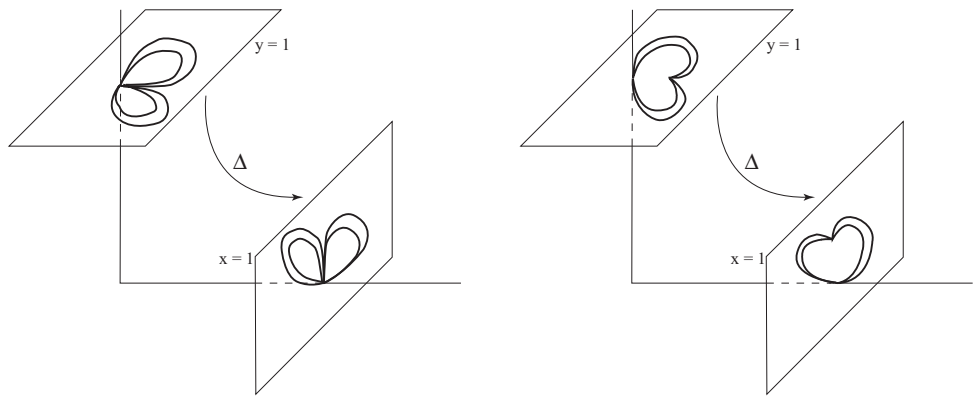
Noting that R_x (resp. R_y) commutes with f_y (resp. f_x), we finally get

$$\Delta \circ (R_x)^{-q} \circ (f_y)^{-p'} = R_y^{p'} \circ f_x^q \circ \Delta. \tag{5.14}$$

Indeed, using (5.12) and (5.13), we have

$$\begin{aligned} \Delta \circ (R_x)^{-q} \circ (f_y)^{-p'} &= (f_x)^q \circ \Delta \circ f_y^{-p'} \\ &= (f_x)^q \circ (R_y)^{p'} \circ \Delta \\ &= R_y^{p'} \circ f_x^q \circ \Delta. \end{aligned} \tag{5.15}$$

Let us now consider a crescent in $\{y = 1\}$ for $f_y^{-p'}$. It is limited by a curve ℓ (the inverse image of a line in Z-space), and its image $\ell_1 = f_y^{-p'} \circ R_x^{-q}(\ell)$. It follows that its



(a) $\epsilon = 0$ (b) $\epsilon \neq 0$

Figure 10: The image of two crescents by Δ

image by Δ yields a crescent limited by $\Delta(\ell)$ and $\Delta(\ell_1)$ (Figure 10). Indeed, the map f_y is approximated by a rotation of angle $-\frac{2\pi q}{p'}$. Hence, $f_y^{-p'}$ behaves like a rotation of angle $2\pi q$. Let $(\boldsymbol{\psi}^0_{x,\epsilon_x}, \boldsymbol{\psi}^\infty_{x,\epsilon_x})$ (resp. $(\boldsymbol{\psi}^0_{y,\epsilon_y}, \boldsymbol{\psi}^\infty_{y,\epsilon_y})$) represent the modulus of f_x (resp. f_y). Then, if we pass to the orbit space by identifying ℓ and ℓ_1 and introducing the spherical coordinate on the orbit space provided by the Fatou coordinate for f_y^{-1}, then the spherical coordinate is transported by Δ on the sphere produced by identifying $\Delta(\ell)$ and $\Delta(\ell_1)$. It is the same as the spherical coordinate induced by the Fatou coordinate for f_x, yielding (5.9). □

Remark 5.4 (i) Since $(\psi^0_{j,\epsilon})'(0) = (\psi^\infty_{j,\epsilon})'(\infty) = \exp(\frac{2\pi^2 a}{q})$, we have

$$(\boldsymbol{\psi}^0_\epsilon)'(0)(\boldsymbol{\psi}^\infty_\epsilon)'(\infty) = \exp(\frac{4\pi^2 a}{q} + \frac{2\pi i m}{q}). \tag{5.16}$$

This is compatible with (5.8) and (5.9). Indeed, let n such that

$$mp' - nq = -1, \tag{5.17}$$

which implies $nq \equiv 1 \pmod{p'}$. Then

$$\begin{aligned}(\boldsymbol{\psi}^0_x)'(0)(\boldsymbol{\psi}^\infty_x)'(\infty) &= \exp(\tfrac{4\pi^2 a_x}{q} + \tfrac{2\pi i m}{q}) = \exp(-\tfrac{4\pi^2 a_y}{p'} + \tfrac{2\pi i}{p'q} + \tfrac{2\pi i m}{q}) \\ &= \exp(-\tfrac{4\pi^2 a_y}{p'} + 2\pi i \tfrac{mp'+1}{p'q}) = \exp(-\tfrac{4\pi^2 a_y}{p'} + 2\pi i \tfrac{n}{p'}) \\ &= ((\boldsymbol{\psi}^0_y)'(0))^{-1}((\boldsymbol{\psi}^\infty_y)'(\infty))^{-1}.\end{aligned} \tag{5.18}$$

(ii) If we were to normalize the representative $(\boldsymbol{\psi}^0_x, \boldsymbol{\psi}^\infty_x)$ of the modulus of f_x, then we would replace it by

$$(\overline{\boldsymbol{\psi}}^0_x, \overline{\boldsymbol{\psi}}^\infty_x) = \left(\boldsymbol{\psi}^0_x \circ L_{\exp(\frac{\pi i m}{q})}, \boldsymbol{\psi}^\infty_x \circ L_{\exp(\frac{\pi i m}{q})}\right),$$

so that

$$(\overline{\boldsymbol{\psi}}^0_x)'(0) = ((\overline{\boldsymbol{\psi}}^\infty_x)'(\infty))^{-1}, \tag{5.19}$$

and, similarly, for f_y^{-1}. If $(\check{\boldsymbol{\psi}}_y^0, \check{\boldsymbol{\psi}}_y^\infty)$ is the normalized modulus for f_y^{-1}, then

$$\begin{cases} (\overline{\boldsymbol{\psi}}_x^0)'(0) = (\check{\boldsymbol{\psi}}_y^0)'(0), \\ (\overline{\boldsymbol{\psi}}_x^\infty)'(\infty) = (\check{\boldsymbol{\psi}}_y^\infty)'(\infty). \end{cases}$$

As expected, the following theorem is true:

Theorem 5.5 *We consider*

- *relatively prime positive integers p' and q,*
- *a germ of analytic function $a_x(\epsilon_x)$,*
- *two germs of family of pairs of analytic diffeomorphisms $(\boldsymbol{\psi}^0_{x,\epsilon_x,\pm}, \boldsymbol{\psi}^\infty_{x,\epsilon_x,\pm})_{\epsilon_x \in V_{x,\pm}(\delta_x)}$, where $V_{x,\pm}(\delta_x)$ are germs of sectors in ϵ as before,*
- *and we suppose that, for $p \equiv -p'$ (mod q), the compatibility condition of Theorem 4.2 is satisfied by $(\boldsymbol{\psi}^0_{x,\epsilon_x,\pm}, \boldsymbol{\psi}^\infty_{x,\epsilon_x,\pm})_{\epsilon_x \in V_{x,\pm}(\delta_x)}$.*

Let ϵ_y, $a_y(\epsilon_y)$ and $(\boldsymbol{\psi}^0_{y,\epsilon_y,\pm}, \boldsymbol{\psi}^\infty_{y,\epsilon_y,\pm})_{\epsilon_y \in V_{y,\pm}(\delta_y)}$ be defined respectively by

$$\begin{cases} \epsilon_y = -\frac{2\pi i q \epsilon_x}{\epsilon_x + 2\pi i p}, \\ \frac{a_y}{p} = -\frac{a_x}{q} + \frac{1}{2\pi i p q}, \\ (\boldsymbol{\psi}^0_{y,\epsilon_y}, \boldsymbol{\psi}^\infty_{y,\epsilon_y}) = (H \circ (\boldsymbol{\psi}^\infty_{x,\epsilon_x})^{-1} \circ H^{-1}, H \circ (\boldsymbol{\psi}^0_{x,\epsilon_x})^{-1} \circ H^{-1}). \end{cases} \tag{5.20}$$

Then, $(\boldsymbol{\psi}^0_{y,\epsilon_y}, \boldsymbol{\psi}^\infty_{y,\epsilon_y})$ satisfies the compatibility condition of Theorem 4.2. Moreover, $a_x \in i\mathbb{R}$ if and only if $a_y \in i\mathbb{R}$.

6 Realization of the unfolding of a resonant diffeomorphism as the unfolding of the holonomy map of a resonant saddle

We consider an unfolding f_ϵ of a codimension 1 resonant diffeomorphism f_0 with multiplier $\exp\left(2\pi i \frac{p}{q}\right)$. We say that it is prepared if its q-th iterate is prepared. In fact, it is not difficult to show that, if g_ϵ is prepared, then the family f_ϵ can be written as

$$f_\epsilon(z) = \exp\left(2\pi i \frac{p}{q}\right)\left(z + \frac{1}{q} z(z^q - \epsilon)(1 + h_1(\epsilon, z))\right). \tag{6.1}$$

The "model" (or formal normal form) for such a family is given by

$$\overline{f}_\epsilon = L_{\exp\left(2\pi i \frac{p}{q}\right)} v_\epsilon^{\frac{1}{q}}, \tag{6.2}$$

where v_ϵ^t is the time t-map of the vector field (2.14). Indeed, it suffices to see that a q-th root of this form exists and to use the uniqueness of a q-th root with a given mutliplier.

Theorem 6.1 *We consider a prepared germ of generic family f_ϵ unfolding a codimension 1 resonant diffeomorphism f_0 with multiplier $\exp\left(2\pi i\frac{p}{q}\right)$. There exists a germ of family of vector fields unfolding a resonant saddle with quotient of eigenvalues $-\frac{p'}{q}$ ($-\frac{p'(1+\eta)}{q}$ in the unfolding) where $p' \in \{1,\dots,q\}$ and $p' \equiv -p \pmod q$, such that the germ of family of holonomies of the x-separatrix on the section $\{x=1\}$ is the family f_ϵ.*

Before going into the proof itself, we first need to refine the preparation.

Theorem 6.2 *We consider a family* (6.1) *and its normal form $\overline{f}_\epsilon$ with same canonical parameter. Then for any $k \in \mathbb{N}^*$ there exists a germ of family of diffeomorphisms h_ϵ tangent to the identity such that*

$$f_\epsilon \circ h_\epsilon - h_\epsilon \circ \overline{f}_\epsilon = O\left(x^{k+1}(x^q-\epsilon)^{k+1}\right). \tag{6.3}$$

Proof. We first consider a change of coordinate $z \mapsto m(z)$ such that

$$f_0 \circ m - m \circ \overline{f}_0 = O\left(x^{k+1}\right).$$

We conjugate f_ϵ by m and obtain a new family. It is of course sufficient to make the proof for this new family, which, for simplicity, we still note by f_ϵ. Let $z_0 = 0$ be the fixed point of f_ϵ, and $z_1, \dots, z_q$ its periodic points. This new family has the property that

$$f_\epsilon^{(\ell)}(z_j) - \overline{f}_\epsilon^{(\ell)}(z_j) = O(\epsilon), \qquad j = 0,\dots,q, \quad \ell \in \{0,\dots,k\}. \tag{6.4}$$

The proof is by induction on k. The case $k=1$ is the preparation already made. We look for (we drop the index ϵ in h_ϵ and f_ϵ)

$$h(z) = z + z^k(z^q-\epsilon)^k P_k(z),$$

where $P_k(z)$ is a polynomial in z of degree less than or equal to k. The polynomial $P_k(z)$ will be uniquely determined if we determine uniquely the $h^{(k)}(z_j)$. These in turn will be found by asking that the k-th derivative of (6.3) vanishes at all fixed points. The k-th derivative of a composition of two functions usually contains many terms. Fortunately, here all derivatives $h^{(\ell)}(z_j) = 0$ for $1 < \ell < k$. Hence, we are left with simple equations. Let $f(z_j) = z_{\sigma(j)}$. Then (we drop the indices)

$$\begin{cases} (f\circ h)^{(k)}(0) = f^{(k)}(0)(h'(0))^k + f'(0)h^{(k)}(0), \\ (f\circ h)^{(k)}(z_j) = f^{(k)}(z_j)(h'(z_j))^k + f'(z_j)h^{(k)}(z_j), \\ (h\circ \overline{f})^{(k)}(0) = h^{(k)}(0)(\overline{f}'(0))^k + h'(0)\overline{f}^{(k)}(0), \\ (h\circ \overline{f})^{(k)}(z_j) = h^{(k)}(z_{\sigma(j)})(\overline{f}'(z_j))^k + h'(z_{\sigma(j)})\overline{f}^{(k)}(z_j). \end{cases}$$

Note that

$$\begin{cases} f'(0) = \overline{f}'(0) = \exp\left(-\frac{\epsilon}{q}\right), \\ f'(z_j) = \overline{f}'(z_j) = \exp\left(2\pi i\frac{p}{q} + \frac{\epsilon}{1+a\epsilon}\right). \end{cases}$$

The set of equations $(f \circ h)^{(k)}(z_j) = (h \circ \overline{f})^{(k)}(z_j)$ in the unknowns $h^{(k)}(z_j)$ has a matrix of the form (up to reordering of rows)

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta^k & -\beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta^k & -\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta^k & -\beta \\ 0 & -\beta & 0 & 0 & \cdots & 0 & \beta^k \end{pmatrix}$$

for nonzero α, β. The determinant of this matrix is equal to $\alpha(\beta^{kq} - \beta^q)$. Hence it does not vanish when $\beta^q \neq 1$ which is the case for nonzero ϵ. Some quantities are small, for instance $\alpha = C_1\epsilon(1 + O(\epsilon))$ and also $\beta^{nq^2} - \beta^q = C_2\epsilon(1 + O(\epsilon))$ for nonzero C_1, C_2 when $k = nq$ for some $n \in \mathbb{N}$. But this is no problem since the right hand sides are also small because of (6.4) and it is possible to find a solution which has a limit for $\epsilon = 0$. □

Proof of Theorem 6.1. To construct the family of vector fields unfolding the resonant saddle, we consider the model family (5.6) in which we forget the index x. For this family the holonomy $\overline{f}_\epsilon$ is exactly the model described above, namely $\overline{f}_\epsilon = L_{\exp\left(2\pi i \frac{p}{q}\right)} v_\epsilon^{\frac{1}{q}}$. The proof is standard and follows closely the corresponding proof in [5]: we first construct the family of vector fields on an abstract manifold and then use the Newlander-Nirenberg (see for instance [9] for the theorem in finite differentiability) to show that this abstract manifold is indeed an open neighborhood of the origin in $\mathbb{C}^2$.

Indeed, we consider $\hat{x}$ in the universal covering of x-space punctured at the origin and a sector

$$\hat{V} = \left\{\hat{x}; |\hat{x}| < 2, \arg \hat{x} \in \left(-\pi, 2\pi + \frac{\pi}{4}\right)\right\}.$$

Let $\mathbb{D}_{r'}$ be a disk in y-space. Over $\hat{V} \times \mathbb{D}_{r'}$ we consider the model family (5.6) (in which we replace x by $\hat{x}$). For $x = 1$, we make the gluing

$$\chi(\hat{x}, y) = (\hat{x}e^{2\pi i}, f_\epsilon \circ (\overline{f}_\epsilon)^{-1}(y))$$

and we extend along the leaves in the obvious way to the domain $\left\{\hat{x}; |\hat{x}| < 2, \arg \hat{x} \in \left(-\pi, \frac{\pi}{4}\right)\right\} \times \mathbb{D}_{r'}$.

A natural almost complex structure can be introduced over this space and shown to be integrable, exactly as in [5]. This allows to fill the hole created by the missing x-axis. The only thing we need to check is that we have sufficient differentiability near $x = 0$. This follows if we have previously applied Theorem 6.2. □

7 Realization of a germ of family unfolding a codimension 1 resonant diffeomorphism

Theorem 7.1 *Let $p, q \in \mathbb{N}^*$ with $(p, q) = 1$ and $p \leq q$, let $a(\epsilon)$ be a germ of holomorphic function at the origin, and let a pair of germs of families of analytic diffeomorphisms*

$$(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})_{\epsilon \in V_\pm}$$

satisfying (5.16) *and the compatibility condition of Theorem* 4.2. *Then there exists a germ of prepared analytic family of diffeomorphisms* f_ϵ *depending on the canonical parameter* ϵ *with the following properties*

- *for* $\epsilon = 0$, f_0 *has a fixed point with multiplier* $\exp\left(2\pi i \frac{p}{q}\right)$;
- *the formal invariant is given by* $a(\epsilon)$;
- *the modulus is given by* $(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})_{\epsilon\in V_\pm}$.

Proof. The proof contains three parts:

(i) **The reduction to the case** $q = 1$**.** Indeed, it follows from Section 5.1 and Theorem 5.5 that it suffices to prove the realization for a germ of family of diffeomorphisms tangent to the identity for $\epsilon = 0$.

(ii) **The local realization.** We first show that we can realize each family

$$(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})_{\epsilon\in V_\pm}$$

in a family of diffeomorphism $f_{\epsilon,\pm}|_{\epsilon\in V_\pm}$ depending analytically on $\epsilon \in V_\pm$, with uniform limit f_0 when $\epsilon \to 0$.

This part is completely analogous to the corresponding part of [1]. Instead of repeating the details, we transform our case to the case studied in [1]. Indeed, suppose we have a prepared family

$$f_\epsilon(z) = z + z(z-\epsilon)[1 + B(\epsilon) + A(\epsilon)z + z(z-\epsilon)h(\epsilon, z)]. \tag{7.1}$$

The change of coordinate and parameter

$$\begin{cases} \epsilon' = \frac{\epsilon^2}{(2+a\epsilon)^2}, \\ z' = \frac{2z-\epsilon}{2+a\epsilon}, \end{cases} \tag{7.2}$$

brings (7.1) to the form studied in [1]

$$\overline{f}_{\epsilon'}(z') = z' + (z'^2 - \epsilon')[1 + B'(\epsilon) + A'(\epsilon)z' + (z'^2 - \epsilon')h'(\epsilon, z')], \tag{7.3}$$

the only difference being that the functions depending analytically on ϵ now depend analytically on $\sqrt{\epsilon'}$. Hence, the two sectors $V_\pm$ in ϵ yield two sectors $V'_\pm$ in ϵ' of opening greater than 2π. Contrary to the case discussed in [1], the two families over V'_- and V'_+ need not be the same.

Each family $(\boldsymbol{\psi}^0_{\epsilon,\pm}, \boldsymbol{\psi}^\infty_{\epsilon,\pm})_{\epsilon\in V_\pm}$ can be transformed into a family depending on $\epsilon' \in V'_\pm$. Hence, it can be realized as the modulus of a family over $V'_\pm$ of the form (7.3) defined over a fixed disk of radius r. Coming back to (z, ϵ), yields two families $f_{\epsilon,\pm}$ of the form (7.1) over $V_\pm$. We need to correct this to a uniform family f_ϵ. This part is what is called the *global realization*.

(iii) **The global realization.**

We have $V_+ \cap V_- = V_l \cup V_r$. The compatibility condition ensures that that the two families are conjugate over the intersection sectors V_l and V_r, by means of analytic diffeomorphisms $h_{\epsilon,l}$ and $h_{\epsilon,r}$ such that

$$h_{\epsilon,j} \circ f_{\epsilon,+} = f_{\epsilon,-} \circ h_{\epsilon,j}, \qquad j \in \{l, r\}.$$

Moreover, as in [1], an appropriate construction of $f_{\epsilon,\pm}$ allows to have

$$h_{\epsilon,j} = id + O\left(\exp\left(-\frac{C}{|\epsilon|}\right)\right),$$

uniformly over $V_\ell \cup V_r$ for some positive constant C.

To correct to a uniform f_ϵ, we construct a uniform f_ϵ over an abstract manifold and we recognize that this manifold is holomorphically equivalent to a neighborhood of the origin in $\mathbb{C}^2$ minus a line corresponding to $\epsilon = 0$. The details are as follows.

The maps $f_{\epsilon,\pm}$ are defined on some opens sets $\mathbb{U}_\pm = \mathbb{D} \times V_\pm$ where $\mathbb{D}$ is a disk of radius r in z-space. The open sets $\{\mathbb{U}_+, \mathbb{U}_-\}$ form an atlas. Let $\mathbb{U}_j = \mathbb{D} \times V_j$, $j \in \{l, r\}$. Over $\mathbb{U}_j$, the transition maps are given by $J_j : \mathbb{U}_j \cap \mathbb{U}_+ \to \mathbb{U}_-$, where

$$J_j = h_{\epsilon,j} \times \mathrm{id}, \qquad (z, \epsilon) \mapsto (h_{\epsilon,j}(z), \epsilon)\,.$$

This is an analytic manifold. Because the limit exists for $\epsilon \to 0$, we can glue in $\mathbb{D} \times \{0\}$. Because of the flatness of $h_{\epsilon,j}$ at $\epsilon = 0$, this yields a C^∞ manifold $\mathcal{M}$. We will endow it with an integrable almost structure and apply Newlander-Nirenberg theorem to recognize that this manifold in an open set in $\mathbb{C}^2$. The construction is completely similar to that of [1], but we include it for purpose of completeness. We call $(z_\pm, \epsilon)$ the coordinates on $\mathbb{U}_\pm$.

We let (Θ_+, Θ_-) be a partition of unity associated to the covering $\{\mathbb{U}_+, \mathbb{U}_-\}$. We can suppose that the derivatives of $\Theta_\pm$ grow no faster than a negative power of the variables. We can also suppose that the $\Theta_\pm$ depend on ϵ alone. Let us first construct a C^∞-diffeomorphism

$$\Omega : \mathcal{M} \to (\mathbb{C}^2, 0) \setminus \{\epsilon = 0\}$$

defined by

$$\Omega = \Theta_+ \cdot (z_+, \epsilon) + \Theta_- \cdot (z_-, \epsilon) = (\Theta_+ z_+ + \Theta_- z_-, \epsilon).$$

Its extension by the identity on $\epsilon = 0$ is again C^∞, because of the flatness of $h_{\epsilon,j}$ at $\epsilon = 0$. This endows $\Omega(\mathcal{M})$ of two complex coordinates (Z, ϵ) where

$$Z = \Theta_+ z_+ + \Theta_- z_-. \tag{7.4}$$

We now show that Ω induces an integrable almost complex structure on $\Omega(\mathcal{M})$. Such an almost complex structure is given by two forms ω, ξ which are $\mathbb{C}$-linear in the sense of this structure.

The almost complex structure is integrable when there exist coordinates (w_1, w_2) such that

$$\langle dw_1, dw_2 \rangle_{\mathbb{C}} = \langle \omega, \xi \rangle_{\mathbb{C}}.$$

In that case, there exists a 2×2 invertible matrix A whose entries are C^∞ functions in (Z, ϵ) such that

$$\begin{pmatrix} \omega \\ \xi \end{pmatrix} = A \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} = A\, dw.$$

In particular, $d \begin{pmatrix} \omega \\ \xi \end{pmatrix} = dA \wedge dw$ contains no $(0,2)$ component. The Newlander-Nirenberg Theorem asserts that this necessary condition is also sufficient for integrability.

For the second form of the complex structure we take $\xi = d\epsilon$. The first form, ω, should play the role of dZ. It will be given by

$$\omega = (\Omega^{-1})^*(\widetilde{\omega}) \tag{7.5}$$

for some form $\widetilde{\omega}$ defined on $\mathcal{M}$. The form $\widetilde{\omega}$ is given by $\widetilde{\omega}_\pm$ on the chart $\mathbb{U}_\pm$. On $\mathbb{U}_-$ we take $\widetilde{\omega}_- = dz_-$. So we want $\widetilde{\omega}_+ = dz_-$ on $\mathbb{U}_+ \cap \mathbb{U}_-$. There, we have

$$\begin{aligned} dz_- &= \tfrac{\partial J_j}{\partial \epsilon} d\epsilon + \tfrac{\partial J_j}{\partial z_+} dz_+ \\ &= \tau_{\epsilon,j} d\epsilon + (1 + T_{\epsilon,j}) dz_+, \end{aligned}$$

where the two functions $\tau_{\epsilon,j}$ and $T_{\epsilon,j}$ are exponentially flat in $|\epsilon|^{-1}$ near $\epsilon = 0$. The gluing is done in the following way: we should remember that δ can been chosen sufficiently small so that J_l (resp. J_r), and then $\tau_{\epsilon,l}$ (resp. $\tau_{\epsilon,r}$) and $T_{\epsilon,l}$ (resp. $T_{\epsilon,r}$) exist for $\arg(\epsilon) \in (-2\delta, +2\delta)$ and $\arg(\epsilon) \in (\pi - 2\delta, \pi + 2\delta)$. We take a C^∞ function $\varphi : \mathbb{R} \to [0,1]$ such that

$$\varphi(x) \equiv \begin{cases} 1, & -\delta < x < \delta, \\ 0, & 2\delta < x < \pi - 2\delta, \\ 1, & \pi - \delta < x < \pi + \delta, \end{cases}$$

and which is decreasing (resp. increasing) in the region $(\delta, 2\delta)$ (resp. $(\pi - 2\delta, \pi - \delta)$). Then,

$$\widetilde{\omega}_+ = \begin{cases} dz_+, & \arg(\epsilon) \in (2\delta, \pi - 2\delta), \\ dz_+ + \varphi(\arg \epsilon)\left(\tau_{\epsilon,r} d\epsilon + T_{\epsilon,r} dz_-\right), & \arg(\epsilon) \in (-\delta, 2\delta), \\ dz_+ + \varphi(\arg \epsilon)\left(\tau_{\epsilon,l} d\epsilon + T_{\epsilon,l} dz_-\right), & \arg(\epsilon) \in (\pi - 2\delta, \pi + \delta). \end{cases}$$

From its construction, the form $\widetilde{\omega} = \widetilde{\omega}_\pm$ on $\mathbb{U}_\pm$ is well defined on $\mathcal{M}$, C^∞ and of type $(1,0)$.

Let us now remark that the difference $\omega - dZ$ decreases exponentially fast as $\epsilon \to 0$. This comes from the fact that $\tau_{\epsilon,j}$ and $T_{\epsilon,j}$, $j \in \{l, r\}$, are exponentially flat in $|\epsilon|^{-1}$ near $\epsilon = 0$. This allows to extend the almost complex structure $\{\omega, d\epsilon\}$ to $\epsilon = 0$, by taking the two forms dz and $d\epsilon$. The resulting almost complex structure is C^∞ in a neighborhood of the origin in $\mathbb{C}^2$.

To show that this complex structure satisfies the necessary condition for integrability we need to show that $\{d\omega, d(d\epsilon)\}$ contains no terms of type $(0,2)$. Obviously $d(d\epsilon) = 0$, so we only need to study $d\omega$. From its construction $d\widetilde{\omega}$ has no terms of type $(0,2)$. The special domain where φ is non identically zero ensures that ω (which is obtained from the pull-back of $\widetilde{\omega}$) also has no term of type $(0,2)$.

Since the almost complex structure satisfies the necessary condition for integrability, we can apply the Newlander-Nirenberg Theorem (for instance [9]) to the manifold $\Omega(\mathcal{M})$. Then, the local charts, which are holomorphic in the sense of this complex structure, are C^∞. Hence, there exists a diffeomorphism $\Gamma : \overline{\Omega(\mathcal{M})} \cap \mathcal{U} \to \mathbb{C}^2$, where $\mathcal{U}$ is a neighborhood of the origin in $\mathbb{C}^2$, which is holomorphic with respect to this structure, and whose image is a neighborhood of the origin in $\mathbb{C}^2$. From the form of the complex structure, it is clear that ϵ can be taken as one of the complex coordinates. So we can suppose that Γ preserves ϵ. The composition $\Gamma \circ \Omega$ is an analytic diffeomorphism between an open set of $\mathcal{M}$ and a neighborhood of the origin in $\mathbb{C}^2$. The map Γ is not unique. We can always choose it in such a way that it sends the curve $z(z-\epsilon) = 0$ to the same curve.

We now conjugate the map (f_ϵ, ϵ) with $\Gamma \circ \Omega$ yielding

$$(g_\epsilon, \epsilon) = (\Gamma \circ \Omega) \circ (f_\epsilon, \epsilon) \circ (\Gamma \circ \Omega)^{-1}.$$

Since g_ϵ is bounded in the neighborhood of $\epsilon = 0$, it is possible to extend it to $\epsilon = 0$ in an analytic way. For each fixed ϵ, the map g_ϵ is conjugated to f_ϵ defined on the slice $\mathcal{M}_\epsilon$. By continuity, it is clear that g_0 is conjugated to $f_0 = \lim_{\epsilon \to 0} f_\epsilon$. □

8 The particular form of the compatibility condition for a resonant saddle of a real vector field

It is easy to verify that, for 2-dimensional vector fields on $\mathbb{C}^2$ coming from the extension of a real vector field on $\mathbb{R}^2$ with a saddle point at the origin, the holonomy maps of the separatrices are reversible, i.e. satisfy

$$\Sigma \circ f = f^{-1} \circ \Sigma, \tag{8.1}$$

where

$$\Sigma(z) = \overline{z} \tag{8.2}$$

is the complex conjugation. For simplicity we only discuss the case of the unfolding of a saddle point with eigenvalues ± 1.

We consider the case of a germ of reversible family of diffeomorphisms satisfying (8.1). To prepare the family to the form (7.1), we need to do a rotation $z \mapsto iz$. Then the reversibility condition becomes

$$\Xi \circ f_\epsilon = (f_{\overline{\epsilon}})^{-1} \circ \Xi, \tag{8.3}$$

where

$$\Xi(z) = -\overline{z}. \tag{8.4}$$

Lemma 8.1 *The formal invariant $a(\epsilon)$ satisfies $a(\overline{\epsilon}) = \Xi(a(\epsilon)) = -\overline{a(\epsilon)}$. In particular $a(\epsilon) \in i\mathbb{R}$ when $\epsilon \in \mathbb{R}$.*

We compare such a family with the time-one map of the vector field

$$v_\epsilon = \frac{z(z-\epsilon)}{1 + a(\epsilon) z}, \tag{8.5}$$

which satisfies

$$\Xi_*(v_\epsilon) = -v_{\overline{\epsilon}} \circ \Xi. \tag{8.6}$$

Proposition 8.2 *(i) It is possible to construct Fatou coordinates such that*

$$\Phi_\epsilon^{\pm} \circ \Xi = \Xi \circ \Phi_{\overline{\epsilon}}^{\mp}.$$

(ii) The modulus is reversible, namely there exist representatives satisfying

$$\Psi_\epsilon^{0,\infty} \circ \Xi = \Xi \circ (\Psi_{\overline{\epsilon}}^{0,\infty})^{-1}. \tag{8.7}$$

(iii) The functions $h_{j,\epsilon,\pm}^{\pm}$, $j \in \{0,1\}$, satisfy

$$\Sigma \circ h_{0,\epsilon,\pm}^{\pm} = h_{1,\overline{\epsilon},\mp}^{\mp} \circ \Sigma.$$

(iv) The compatibility condition becomes: for $\epsilon \in \mathbb{R}$,

$$\psi_\epsilon^G = \Sigma \circ (\psi_\epsilon^G)^{-1} \circ \Sigma.$$

Let us explain this in words. We know that if we have a family of real vector fields then, in adequate coordinates, the Glutsyuk modulus is reversible. Hence, if we start with families of germs of diffeomorphisms $(\Psi_{\epsilon,\pm}^{0,\infty})_{\epsilon \in V_{\delta,\pm}}$ which are 1-summable in ϵ and reversible, it turns out that the compatibility condition is exactly equivalent to the reversibility of the Glutsyuk modulus for $\epsilon \in \mathbb{R}$ when derived from $(\Psi_{\epsilon,\pm}^{0,\infty})_{\epsilon \in V_{\delta,\pm}}$.

Proof of Proposition 8.2.

(i) and (ii) are obvious.

(iii) This follows from the fact that $E \circ \Xi = \Sigma \circ E$.

(iv) We derive Glutsyuk moduli for $\epsilon \in V_l^{\pm}$:

$$\begin{cases} \psi_{\epsilon,+}^G = h_{1,\epsilon,+}^{+} \circ (h_{0,\epsilon,+}^{+})^{-1}, \\ \psi_{\epsilon,-}^G = h_{1,\epsilon,-}^{-} \circ (h_{0,\epsilon,-}^{-})^{-1}. \end{cases}$$

Hence,

$$\psi_{\epsilon,+}^G = \Sigma \circ (\psi_{\overline{\epsilon},-}^G)^{-1} \circ \Sigma.$$

We also know, by the uniqueness of the Glutysuk modulus that, for $\epsilon \in \mathbb{R}$, there exists $a, b \in \mathbb{C}^*$ such that $\psi_{\epsilon,+}^G = L_a \circ \psi_{\epsilon,-}^G \circ L_b$.

We first prove that $b = \overline{a}$. Indeed,

$$\psi_{\epsilon,+}^G = \Sigma \circ L_b \circ (\psi_{\epsilon,+}^G)^{-1} \circ L_a \circ \Sigma. \tag{8.8}$$

Also,

$$\begin{aligned} \psi_{\epsilon,+}^G &= L_a \circ \psi_{\epsilon,-}^G \circ L_b \\ &= L_a \circ \Sigma \circ (\psi_{\epsilon,+}^G)^{-1} \circ \Sigma \circ L_b \\ &= \Sigma \circ L_{\overline{a}} \circ (\psi_{\epsilon,+}^G)^{-1} \circ L_{\overline{b}} \circ \Sigma. \end{aligned} \tag{8.9}$$

Comparing (8.8) and (8.9) yields $b = \overline{a}$ when $\psi^G_{\epsilon,+} \neq id$. (Note that when $\psi^G_{\epsilon,+} = id$, then $\psi^G_{\epsilon,+}$ is symmetric and we are finished.)

So, for $\epsilon \in \mathbb{R}$, we let $\psi^G_\epsilon = L_{a^{1/2}} \circ \psi^G_{\epsilon,-} \circ L_{\overline{a}^{1/2}}$. Then $\psi^G_\epsilon = \Sigma \circ (\psi^G_\epsilon)^{-1} \circ \Sigma$. Indeed,

$$\begin{aligned}
\Sigma \circ (\psi^G_\epsilon)^{-1} \circ \Sigma &= \Sigma \circ L_{\overline{a}^{-1/2}} \circ (\psi^G_{\epsilon,-})^{-1} \circ L_{a^{-1/2}} \circ \Sigma \\
&= L_{a^{-1/2}} \circ \Sigma \circ (\psi^G_{\epsilon,-})^{-1} \circ \Sigma \circ L_{\overline{a}^{-1/2}} \\
&= L_{a^{-1/2}} \circ \psi^G_{\epsilon,+} \circ L_{\overline{a}^{-1/2}} \\
&= \psi^G_\epsilon.
\end{aligned}$$

□

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References

[1] C. Christopher and C. Rousseau, The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point, preprint CRM (2008), ArXiv:0809.2167.

[2] J. Ecalle, *Les fonctions résurgentes*, Publications mathématiques d'Orsay, 1985.

[3] A. A. Glutsyuk, Confluence of singular points and nonlinear Stokes phenomenon, *Trans. Moscow Math. Soc.*, **62** (2001), 49–95.

[4] Y. Ilyashenko, Nonlinear Stokes phenomena, in *Nonlinear Stokes phenomena*, Y. Ilyashenko editor, Advances in Soviet Mathematics, vol. 14, Amer. Math. Soc., Providence, RI, (1993), 1-55.

[5] Y. Ilyashenko and S. Yakovenko, *Lectures on Analytic Differential Equations*, Graduate Studies in Mathematics, vol. 86, Amer. Math. Soc., Providence, RI, 2008.

[6] F. Loray, Cinq leçons sur la structure transverse d'une singularité de feuilletage holomorphe en dimension 2 complexe, *Monographies Red TMR Europea Sing. Ec. Dif. Fol.* **1** (1999), 1–92.

[7] P. Mardešić, R. Roussarie and C. Rousseau, Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms, *Moscow Mathematical Journal*, **4** (2004), 455–502.

[8] J. Martinet and J.-P. Ramis, Classification analytique des équations différentielles non linéaires résonantes du premier ordre, *Ann. Sci. École Norm. Sup.*, **16** (1983), 571–621.

[9] A. Nijenhuis and W.B. Woolf, Some integration problems in almost-complex and complex manifolds, *Annals of Mathematics*, **77** (1963), 424–489.

[10] R. Oudkerk, The parabolic implosion for $f_0(z) = z+z^{\nu+1}+O(z^{\nu+2})$, thesis, University of Warwick (1999).

[11] R. Pérez-Marco and J.-C. Yoccoz, Germes de feuilletages holomorphes à holonomie prescrite, *Astérisque*, **222** (1994), 345–371.

[12] C. Rousseau, Normal forms, bifurcations and finiteness properties of vector fields, in *Normal forms, bifurcations and finiteness properties of vector fields*, NATO Sci. Ser. II Math. Phys. Chem., 137, Kluwer Acad. Publ., Dordrecht, 2004, 431–470.

[13] C. Rousseau and C. Christopher, Modulus of analytic classification for the generic unfolding of a codimension one resonant diffeomorphism or resonant saddle, *Annales de l'Institut Fourier*, **57** (2007), 301–360.

[14] M. Shishikura, Bifurcations of parabolic fixed points, in *The Mandelbrot set, theme and variations*, Tan Lei Editor, London Math. Soc. Lecture Note Ser., **274**, Cambridge Univ. Press, Cambridge, 2000, 325–363.

[15] S. M. Voronin, A. A. Grintchy, An analytic classification of saddle resonant singular points of holomorphic vector fields in the complex plane, *J. Dynam. Control Syst.*, **2** (1996), 21–53.