The root extraction problem*

C. Rousseau, DMS and CRM, Université de Montréal

January 2006

Abstract

The N-th root extraction problem for germs of diffeomorphisms $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is the problem of finding a germ of diffeomorphism $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $g^N = f$, where g^N is the N-th iterate of g under composition. Depending on f and on the multiplier of g at the origin there can be formal and analytic obstructions to a solution of the problem. By considering an unfolding of f we explain these obstructions. Indeed each analytic obstruction corresponds to an accumulation of periodic points which, in turn, are an obstruction to taking an N-th root of the unfolding. We apply this to the problem of the section of a curvilinear angle in N equal parts in conformal geometry.

1 Introduction

We consider a germ of diffeomorphism $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with $f'(0) = \lambda$. The classical N-th root extraction problem for f is the problem of finding a germ of analytic diffeomorphism $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that $g^N = f$, where $g^N = \underbrace{g \circ g \circ \cdots \circ g}_{N}$. When $|\lambda| \neq 1$, the problem is solvable as f is linearizable, i.e. there exists a germ of holomorphic diffeomorphism $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h \circ f \circ h^{-1}(z) = \lambda z$. Then we find N functions g_j which are N-th roots of f given by $g_j(z) = h^{-1}(\nu_j h(z))$, $j = 0, \dots, N-1$, where $\nu_0, \dots, \nu_{N-1}$ are the N-th roots of λ. When $|\lambda| = 1$ the same occurs as soon as f is linearizable. This is always the case when $\lambda = \exp(2\pi i\alpha)$ with α irrational diophantian, more precisely when α satisfies the Brujno condition. When α is irrational Liouvillian, more precisely α does not satisfy the Brujno condition, then a formal solution h exists, but, generically, small denominators are an obstruction to convergence. This does not exclude a priori the existence of an N-th root. Indeed it is shown in Pfeiffer [12] that there exists a non linearizable f with $f'(0) = \exp(2\pi i\alpha)$ which has a square root. Such an example is simply constructed by taking some non linearizable g with $g'(0) = \exp(\pi i\alpha)$ and taking $f^2 = g$. In the same paper Pfeiffer constructs examples of maps which have no square root, by showing the divergence of the formal series of their square roots.

In this paper we consider the case where λ is a root of unity: $\lambda = \exp(2\pi i\frac{p}{q})$. It is clear that if a germ f admits an N-th root, then any germ $\tilde{f}$ conjugate to f admits an N-th root, so this is really a property of the equivalence class of f under conjugacy. Let us first recall the known results (for instance [4], [3]). The map f is linearizable if and only if $f^q = id$,

*This work is supported by NSERC in Canada.

which occurs only exceptionally. In general $f^q(z) = z + Cz^{kq+1} + o(z^{kq+1})$ with $C \neq 0$. We first look for formal N-th roots $\hat{g}$ of f with $\hat{g}'(0) = \mu_j = \exp(2\pi i \frac{p+jN}{qN})$, $j = 0, \dots, N-1$. Depending on j there may exist some formal obstructions to find $\hat{g}$. Moreover, when $\hat{g}$ exists it converges very exceptionally. The conditions for the convergence of $\hat{g}$ can be read on the Ecalle-Voronin modulus of f. A natural question is to ask why the existence of an N-th root is so exceptional.

In this paper we give a geometric explanation of this phenomenon for the codimension 1 case. This is done through unfolding the diffeomorphism f in a family f_ϵ. In the unfolding we observe accumulation of periodic points for f_ϵ. Their presence is an obstruction to the N-th root extraction problem.

We apply this to a problem in conformal geometry, namely the problem of the section of a curvilinear angle in N equal parts. Curvilinear angles are given by two germs of arcs of real analytic curves in $\mathbb{C}$ and we consider the conformal equivalence of curvilinear angles. Each germ of curve determines a germ of Schwarz reflection. In the case of the real axis, its associated Schwarz reflection is $\Sigma(z) = \overline{z}$. It preserves the size of angles and reverses their sign. If (γ, z_0) is any germ of real algebraic curve at a point $z_0 \in \mathbb{C}$, let h be an analytic map sending it to $(\mathbb{R}, 0)$. Then its associated Schwarz reflection is $\Sigma_1 = h^{-1} \circ \Sigma \circ h$. ($\Sigma_1$ is an involution reversing angles and with γ as set of fixed points). The composition of the two germs of Schwarz reflections associated to the two arcs of a curvilinear angle is a germ of analytic diffeomorphism f with a fixed point which has a symmetry property with respect to the Schwarz reflections: if Σ_j is any of the Schwarz reflections associated to one of the two curves we have $f \circ \Sigma_j = \Sigma_j \circ f^{-1}$.

The symmetry property $f \circ \Sigma = \Sigma \circ f^{-1}$, for $\Sigma(z) = \overline{z}$, is exactly the symmetry property of the holonomy of a separatrix of a saddle point of a real vector field. Also the holonomy of the strong separatrix of a saddle-node of a real vector field has this property, which is studied in detail in [2].

The whole paper is limited to study the codimension 1 phenomenon. The paper is organized as follows. In Section 2 we recall the modulus of analytic classification of a family unfolding a germ of resonant diffeomorphism and the condition for solving the root extraction problem for f. In Section 3 we discuss the renormalized maps for f_ϵ and their link with the localization of f_ϵ at its periodic points. In Section 4 we explain the obstruction to the root extraction problem. In Section 5 we make the link with the problem of section of a curvilinear angle in N equal parts in the conformal geometry of germs of curvilinear angles.

2 Preliminaries

2.1 Modulus of the unfolding of a resonant diffeomorphism

We briefly recall the results of [14] to which the reader can refer for more details. We consider a germ of one-parameter family of diffeomorphisms f_ϵ unfolding a germ of resonant diffeomorphism $f_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$f_0(z) = \exp\left(2\pi i \frac{p}{q}\right) z + \frac{z^{q+1}}{q} + o(z^{q+1}). \tag{2.1}$$

In the case $q > 1$ such a family will have the form

$$f_\epsilon(z) = \exp\left(2\pi i \frac{p}{q} + \eta(\epsilon)\right) z + \frac{z^{q+1}}{q} + o(z^{q+1}) + O(\epsilon). \tag{2.2}$$

We will also use this form when $p = q = 1$ and there is a constraint forcing $z = 0$ to remain a fixed point. This is the case for instance when we consider the holonomy of a saddle point: the separatrix remains a fixed point of the unfolding. Otherwise, when $p = q = 1$ we consider an unfolding

$$f_\epsilon(z) = z - \eta(\epsilon) + z^2 + o(z^2) + O(\epsilon). \tag{2.3}$$

We consider generic families f_ϵ unfolding (2.1). For the family (2.2) (resp. (2.3)) the genericity condition is given by $\frac{\partial^2 f_\epsilon}{\partial z \partial \epsilon} \neq 0$ (resp. $\frac{\partial f_\epsilon}{\partial \epsilon} \neq 0$).

Case of a family (2.2)**.** To describe the modulus of a family f_ϵ of the form (2.2) is equivalent to describe the modulus of the family f_ϵ^q. We now limit ourselves to the discussion of the family f_ϵ^q. Modulo a "preparation" (i.e. an analytic change of coordinate and parameter) we can always suppose that its fixed points are given by $z(z^q - \epsilon) = 0$. To describe its modulus, the point of view is to compare the family with the time-one map of the vector field

$$\frac{z(z^q - \epsilon)}{1 + a(\epsilon) z^q} \frac{\partial}{\partial z}, \tag{2.4}$$

which we call the "model family" in the case of (2.2). The diffeomorphism can be conjugated to the model family on some adequate sectorial domains in (z, ϵ)-space. For fixed ϵ the modulus measures the obstruction to a conjugacy over a full neighborhood of the origin in z-space.

Using a change of coordinate and parameter it is possible to "prepare" the family f_ϵ, i.e. to bring it to the form:

$$f_\epsilon^q(z) = z + z(z^q - \epsilon)(1 + A(\epsilon) + (z^q - \epsilon) h(z, \epsilon)), \tag{2.5}$$

so that the fixed points z_j of f_ϵ^q (i.e. the fixed and periodic points of f_ϵ) coincide with the singular points of (2.4) and that their multipliers λ_j be equal to $\exp(\mu_j)$, where μ_j are the eigenvalues of (2.4) at the z_j.

In a prepared family the parameter ϵ (called the *canonical parameter*) is an analytic invariant. Hence a conjugacy between two prepared families must preserve their canonical parameters. The meaning of the formal invariant $a(\epsilon)$ is obtained through the following property

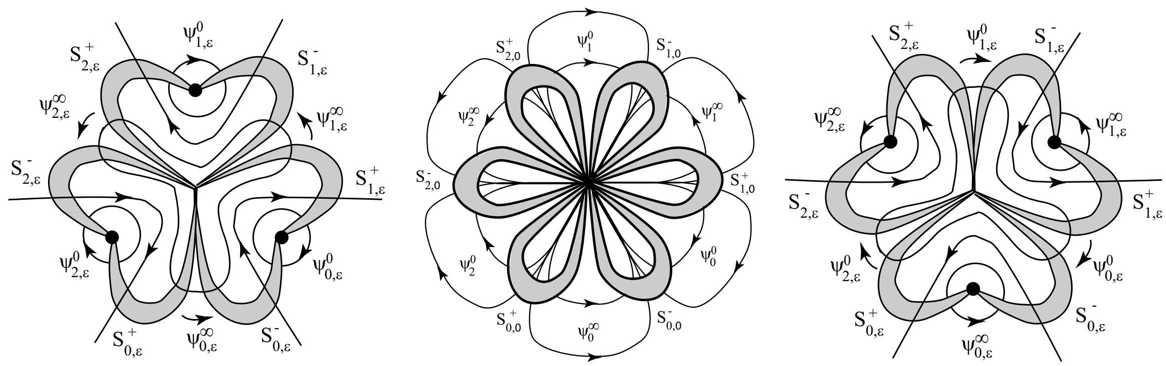
$$a(\epsilon) = \sum_j \frac{1}{\mu_j}. \tag{2.6}$$

This yields the following relation between $A(\epsilon)$ and $a(\epsilon)$

$$a(\epsilon) = \frac{q}{\ln(1 + q\epsilon(1 + A(\epsilon)))} + \frac{1}{\ln(1 - \epsilon(1 + A(\epsilon)))}. \tag{2.7}$$

In general we consider all values of ϵ in a neighborhood of the origin. For the phenomena described below we will mostly limit ourselves to values of ϵ for which at least one of the λ_j satisfies $|\lambda_j| = 1$ (the Siegel domain).

To compare f_ϵ^q with the corresponding model diffeomorphism, we compare their orbit spaces. The orbit space of f_ϵ^q is obtained by taking $2q$ curves $l_j^\pm$, $j = 0, \dots, q-1$, and their images $f_\epsilon^q(l_j^\pm)$ as in Figure 1. The curves $l_j^\pm$ and their images determine crescents $S_{j,\epsilon}^\pm$.



(a) Negative Siegel direction (b) $\epsilon = 0$ (c) Positive Siegel direction

Figure 1: The maps $\psi_{j,\epsilon}^{0,\infty}$ for $q = 3$, $a(\epsilon) \in i\mathbb{R}$ and ϵ in Siegel direction

Passing to the orbit space, we identify $l_j^\pm$ and $f_\epsilon^q(l_j^\pm)$. The corresponding space has the conformal structure of a sphere ($\mathbb{CP}^1$): we will denote it by $\mathbb{S}_{j,\epsilon}^\pm$. The fixed points z_j of f_ϵ^q correspond to the distinguished points 0 and ∞ on the spheres. The $2q$ spheres are necessary to cover the orbit space of f_0^q but some orbits have representatives in different spheres. So it is necessary to identify the points in different spheres corresponding to the same orbit. This is done through the germs of holomorphic maps $\psi_{j,\epsilon}^0$ (resp. $\psi_{j,\epsilon}^\infty$) defined respectively in the neighborhood of 0 (resp. ∞) on the spheres. The coordinates on the spheres are given up to linear maps (which are the only global holomorphic diffeomorphisms of $\mathbb{CP}^1$ fixing 0 and ∞). It is possible to choose the coordinates so that

$$(\psi_{j,\epsilon}^0)'(0) = (\psi_{j,\epsilon}^\infty)'(\infty) = \exp\left(\frac{2\pi^2 a}{q}\right). \tag{2.8}$$

So far we have described the modulus space of f_ϵ^q. But each orbit of f_ϵ^q represents q orbits of f_ϵ. This is reflected in the fact that only two $\psi_{j,\epsilon}^{0,\infty}$ are independent and the others are related through:

$$\psi_{\sigma(j)}^{0,\infty} = B^{-1} \circ \psi_j^{0,\infty} \circ B, \tag{2.9}$$

where σ is the permutation of $\{0, \dots, q-1\}$ given by

$$\sigma(j) \equiv j + p \,(\mathrm{mod}\; q) \tag{2.10}$$

and

$$B(w) = \exp\left(\frac{2\pi i}{q}\right) w. \tag{2.11}$$

Theorem 2.1 *[14] The complete modulus of analytic classification of the family f_ϵ^q (and hence f_ϵ) is given by $a(0)$, together with the family of equivalent classes of 2-tuples*

$$[(\psi^0_{0,\epsilon}, \psi^\infty_{0,\epsilon})]/\sim, \tag{2.12}$$

where the equivalence relation is defined by :

$$(\psi^0_{0,\epsilon}, \psi^\infty_{0,\epsilon}) \sim (\overline{\psi}^0_{0,\epsilon}, \overline{\psi}^\infty_{0,\epsilon}) \Longleftrightarrow \exists c \in \mathbb{C}^* \quad \overline{\psi}^{0,\infty}_{0,\epsilon}(w) = c^{-1}\psi^{0,\infty}_{0,\epsilon}(cw). \tag{2.13}$$

Remark 2.2 It is not possible to define the $\psi^{0,\infty}_{j,\epsilon}$ depending continuously on ϵ on a neighborhood of the origin. It is however possible to cover a neighborhood of the origin in ϵ-space with two sectors $V_\pm$, and to choose families $(\psi^{0,\infty}_{j,\epsilon,\pm})|_{\epsilon\in V_\pm}$ depending analytically on $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$ (details in [14]), where

$$\begin{aligned} V_+ &= \{\epsilon; |\epsilon| < \rho, \arg(\epsilon) \in (-\pi/2 + \delta, 3\pi/2 - \delta)\\ V_- &= \{\epsilon; |\epsilon| < \rho, \arg(\epsilon) \in (-3\pi/2 + \delta, \pi/2 - \delta). \end{aligned} \tag{2.14}$$

δ can be chosen arbitrarily small and ρ depends on δ. For the rest of the paper we drop the lower indices $\pm$.

Definition 2.3 We call the *Siegel direction of the origin* in parameter space ϵ the direction where $|\lambda_0| = 1$. As $\lambda_0 = \exp(-\epsilon)$ this yields $\epsilon \in i\mathbb{R}$. The *Siegel direction of the periodic orbit* is the direction where $|\lambda_j| = 1$, i.e. $\frac{q\epsilon}{1+a\epsilon} \in i\mathbb{R}$. When $a \in i\mathbb{R}$ both coincide. The negative (resp. positive) Siegel direction of the origin is the half part of the Siegel direction of the origin contained in V_- (resp. V_+). The negative (resp. positive) Siegel direction of the periodic point is the half part of the Siegel direction of the periodic orbit contained in V_- (resp. V_+).

Although we do not want to reproduce the proof of Theorem 2.1 we will need later the following tools introduced in the proof.

We use a change of coordinate

$$z = p_\epsilon(Z) = \begin{cases} \left(\frac{\epsilon}{1-e^{q\epsilon Z}}\right)^{1/q} & \epsilon \neq 0\\ \left(-\frac{1}{qZ}\right)^{1/q} & \epsilon = 0, \end{cases} \tag{2.15}$$

conjugating f_ϵ^q with F_ϵ^q which is a small perturbation of the translation by 1. We consider $2q$ translation domains $Q^\pm_{j,\epsilon}$, $j = 0, \dots, q-1$, in Z-space (see Figure 2) on which there exists a change of coordinate $W = \Phi^\pm_{j,\epsilon}$ conjugating F_ϵ^q with the translation by 1:

$$\Phi^\pm_{j,\epsilon}(F_\epsilon^q(Z)) = \Phi^\pm_{j,\epsilon}(Z) + 1. \tag{2.16}$$

The maps $\Phi^\pm_{j,\epsilon}$ are called the *Fatou coordinates*. As F_ϵ commutes with F_ϵ^q they satisfy

$$\Phi^\pm_{\sigma(j),\epsilon} \circ F_\epsilon = T_{\frac{1}{q}} \circ \Phi^\pm_{j,\epsilon}, \tag{2.17}$$

where $T_{\frac{1}{q}}$ is the translation $W \mapsto W + \frac{1}{q}$ and σ is defined in (2.10).

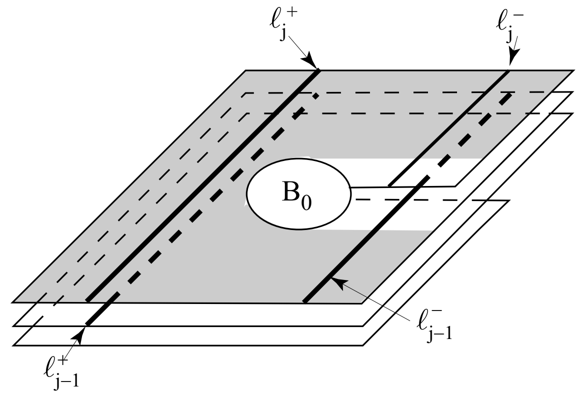


Figure 2: The Z-space. The shaded area is a translation domain.

The $2q$ coordinates on the $2q$ spheres $\mathbb{S}_{j,\epsilon}^{\pm}$ discussed before are then obtained (as functions of z) by means of $E \circ \Phi_{j,\epsilon}^{\pm} \circ p_\epsilon^{-1}$, where $E(W) = \exp(-2\pi i W)$. The lifting of the $\psi_{j,\epsilon}^{0,\infty}$ in W-space are obtained as

$$\begin{cases} \Psi_{j,\epsilon}^{\infty} = \Phi_{j,\epsilon}^{-} \circ (\Phi_{j,\epsilon}^{+})^{-1} \\ \Psi_{j,\epsilon}^{0} = \Phi_{j,\epsilon}^{-} \circ (\Phi_{j+1,\epsilon}^{+})^{-1}, \end{cases} \tag{2.18}$$

$j = 0, \dots, q-1$, where indices are (mod q). The relation (2.17) yields (2.9).

Remark 2.4 Note on the choice of the indices 0 and ∞ in the functions $\psi_{j,\epsilon}^{0,\infty}$ defining the modulus. The direction of the maps $\psi_{j,\epsilon}^{0,\infty}$ follow the dynamics of f_ϵ^q. When the parameter is in the Siegel direction then the map $\psi_{j,\epsilon}^{0}$ (resp. $\psi_{j,\epsilon}^{\infty}$) goes clockwise (resp. counterclockwise).

The case of family (2.3)**.** The model family in that case is the time-one map of the vector field

$$\frac{z^2-\epsilon}{1+a(\epsilon)z}\,\frac{\partial}{\partial z}. \tag{2.19}$$

It is possible to prepare the family to a form

$$f_\epsilon(z) = z + (z^2-\epsilon)(1 + A(\epsilon) + (z^2-\epsilon)h(z,\epsilon)). \tag{2.20}$$

As before we prepare the family so that the parameter becomes an analytic invariant. We then introduce a change of coordinate analogous to (2.15)

$$z = \begin{cases} \sqrt{\epsilon}\frac{1+e^{2\sqrt{\epsilon}Z}}{1-e^{2\sqrt{\epsilon}Z}} & \epsilon \neq 0 \\ -\frac{1}{Z} & \epsilon = 0. \end{cases} \tag{2.21}$$

The rest of the analysis, including the construction of Fatou coordinates, is similar to the previous case and we replace ϵ by $\sqrt{\epsilon}$ in the definition of $V_\pm$ in (2.14). All details appear in [8]. The crescents and maps $\psi_\epsilon^{0,\infty}$ appear in Figure 3.

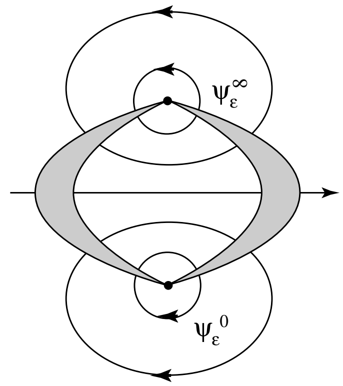


Figure 3: The maps $\psi_\epsilon^{0,\infty}$ for (2.20), $a(\epsilon) \in i\mathbb{R}$ and $\epsilon \in \mathbb{R}^-$

2.2 The Martinet-Ramis point of view for the modulus

Although the paper [7] is primarily concerned with the modulus of a resonant saddle, the authors also treat the modulus of a resonant diffeomorphism. Instead of using $2q$ spheres to describe the modulus, and $2q$ germs of diffeomorphisms, of which only 2 are independent, they use only two spheres and two germs of diffeomorphisms. We call this the Martinet-Ramis point of view.

In the Martinet-Ramis point of view we can see the modulus as a pair of germs of maps $(\psi_{0,\epsilon}^0, \tilde{\psi}_\epsilon^\infty)$ from $\mathbb{S}_{1,\epsilon}^+$ to $\mathbb{S}_{0,\epsilon}^-$. The map $\tilde{\psi}_\epsilon^\infty$ identifies points belonging to the same orbit. As $w_1 \in \mathbb{S}_{j,\epsilon}^-$ and $w_2 \in \mathbb{S}_{\sigma^m(j),\epsilon}^-$ belong to the same orbit if $w_2 = \exp(\frac{-2\pi i m}{q})w_1 = \ell(w_1)$ we need to take m such that $mp \equiv -1 \pmod q$. Then

$$\tilde{\psi}_\epsilon^\infty = \ell \circ \psi_{1,\epsilon}^\infty = \psi_{0,\epsilon}^\infty \circ \ell \tag{2.22}$$

where

$$\ell(w) = \exp\left(-\frac{2\pi i m}{q}\right) w, \qquad mp \equiv -1 \pmod q. \tag{2.23}$$

If we let $\psi_{0,\epsilon}^0 = \tilde{\psi}_\epsilon^0$ we will denote the modulus by $(\tilde{\psi}_\epsilon^0, \tilde{\psi}_\epsilon^\infty)$. When the context is clear we will simply denote it by $(\psi_\epsilon^0, \psi_\epsilon^\infty)$.

As $(\psi_{j,\epsilon}^0)'(0) = (\psi_{j,\epsilon}^\infty)'(\infty) = \exp(\frac{2\pi^2 a}{q})$ this yields

$$(\tilde{\psi}_\epsilon^0)'(0)(\tilde{\psi}_\epsilon^\infty)'(\infty) = \exp\left(\frac{4\pi^2}{q} + \frac{2\pi i m}{q}\right). \tag{2.24}$$

Note that (2.24) remains valid under global changes of coordinates on the spheres $\mathbb{S}_{1,\epsilon}^+$ and $\mathbb{S}_{0,\epsilon}^-$ preserving 0 and ∞.

2.3 The root extraction problem for f

Let us first discuss the case $f'(0) = 1$ and

$$f(z) = z + z^2 + o(z^2). \tag{2.25}$$

The following lemma, which looks trivial, contains the idea which will be used in the further explanations of the analytic obstructions to the N-th root problem.

Lemma 2.5 *Let f be as in* (2.25) *and*

$$g(z) = \exp\left(2\pi i \frac{j}{N}\right) z + o(z), \tag{2.26}$$

for $j = 0, \dots, N-1$. Then an N-th root g of f with $N > 1$ necessarily has the form $g(z) = z + o(z)$, i.e. $j = 0$.

Proof: If $g^N = f$ then necessarily $g^N(z) = z + O(z^{N+1})$ as soon as $j \neq 0$. □

Remark 2.6 (1) The formal obstruction for solving $g^N = f$ when $j \neq 0$ can easily be understood. Indeed f has a double fixed point at the origin, while g, if it exists, has a single one. So any unfolding of f will have two fixed points. Let g_ϵ be any unfolding of g with $g_0 = g$ given in (2.26), yielding that $g_\epsilon^N = f_\epsilon$ is an unfolding of f. Then g_ϵ has a unique fixed point and the other fixed point of f_ϵ corresponds to a periodic point z_1 of g_ϵ of period N. The orbit of z_1 is given by $z_1, \dots, z_N$, with $z_{j+1} = g_\epsilon(z_j)$. But then all z_j are fixed points of f_ϵ, a contradiction. In the limit for $\epsilon = 0$, in order that $f = g^N$, where g is as in (2.26) with $j \neq 0$, then the origin must be a fixed point of f of multiplicity $N + 1$ as it should be the coallescence of a fixed point of f with a periodic point of multiplicity N.

(2) The simple explanation of (1) is very important. We will see the same phenomenon being reproduced in cascades. These cascades will explain the analytic obstructions to the root extraction problem.

Theorem 2.7 *([3] and [4]) Let f be as in* (2.25). *Then f has an N-th root g with $N > 1$ of the form $g(z) = z + o(z)$ if and only if the maps $\psi^{0,\infty}$ of the Ecalle-Voronin modulus of f satisfy*

$$\psi^{0,\infty}(w) = w\xi^{0,\infty}(w^N), \tag{2.27}$$

for some germs of non vanishing analytic functions $(\xi^0, 0)$ and (ξ^∞, ∞).

Idea of the proof. (To complete to a full proof see corresponding proof for the unfolding in Theorem 4.1 (2)). The orbit space of f is given by the two spheres identified in the neighborhood of 0 and ∞ via ψ^0 and ψ^∞ respectively. We want to describe the dynamics on the orbit space. Then, on each sphere, the action of f can be viewed as the time-one map of the vector fields $\dot{w} = -2\pi i w$, i.e. the identity $w \mapsto w$. On each sphere an N-th root of f is given by the time $1/N$ of the same vector fields, namely $w \mapsto L(w) = \exp(-2\pi i/N)w$. The N-th roots on the two spheres must be compatible with $\psi^{0,\infty}$, i.e. $\psi^{0,\infty}$ must commute with L. This is equivalent to (2.27). □

Corollary 2.8 *Let f be as in* (2.25) *such that its modulus satisfies* (2.27) *for $N > 1$ and let (ψ^0, ψ^∞) be its Ecalle-Voronin modulus. Then the Ecalle-Voronin modulus of the N-th root g tangent to the identity has a modulus of the form $(\tilde{\psi}^0, \tilde{\psi}^\infty)$ with*

$$\tilde{\psi}^{0,\infty} = R \circ \psi^{0,\infty} \circ R^{-1}, \tag{2.28}$$

where $R(w) = w^N$. Hence $\tilde{\psi}^{0,\infty}(w) = w(\xi^{0,\infty}(w))^N$.

PROOF. Let $\tilde{\Phi}^{\pm}$ be the Fatou coordinates of g. Then the Fatou coordinates $\Phi^{\pm}$ of f are given by $\Phi^{\pm} = \frac{1}{N}\tilde{\Phi}^{\pm} = L_{1/N} \circ \tilde{\Phi}^{\pm}$, where $L_{\alpha}(W) = \alpha W$ (since the Fatou coordinates are unique up to translation). Hence the modulus of g is given by

$$\tilde{\Psi} = \tilde{\Phi}^{-} \circ (\tilde{\Phi}^{+})^{-1} = L_N \circ \Phi^{-} \circ (\Phi^{+})^{-1} \circ L_{1/N}. \tag{2.29}$$

Let $E(W) = \exp(-2\pi i W)$. It follows that

$$\tilde{\psi} = E \circ \tilde{\Psi} \circ E^{-1} = R \circ E \circ \Psi \circ E^{-1} \circ R^{-1}. \tag{2.30}$$

□

The theorem 2.7 can be generalized as follows.

Theorem 2.9 *([3] and [4]) Let*

$$f(z) = \exp\left(2\pi i \frac{p}{q}\right) z + z^{q+1} + o(z^{q+1}) \tag{2.31}$$

and let $N|p$, i.e. $p = Np'$. There exists a germ of map $g(z) = \exp\left(2\pi i \frac{p'}{q}\right) z + o(z)$ such that $g^N = f$ if and only if the components of the modulus $\psi_j^{0,\infty}$, $j = 0, \dots, q-1$, satisfy

$$\psi_j^{0,\infty}(w) = \exp\left(-\frac{2\pi i}{N}\right) \psi_j^{0,\infty}\left(\exp\left(\frac{2\pi i}{N}\right) w\right), \quad j = 0, \dots, q-1, \tag{2.32}$$

i.e. $\psi_j^{0,\infty}(w) = w\xi_j^{0,\infty}(w^N)$ for some non vanishing germs of maps $\xi_j^{0,\infty}$.

Corollary 2.10 *Let f be as in* (2.31) *which satisfies* (2.32) *for $N|p$ and let $[(\psi_0^0, \psi_0^\infty)]/\sim$ be its Ecalle-Voronin modulus. Then the modulus of the N-th root g of f with multiplier $\exp(2\pi i \frac{p'}{q})$ has the form $[(\tilde{\psi}_0^0, \tilde{\psi}_0^\infty)]/\sim$ with*

$$\tilde{\psi}_0^{0,\infty}(w) = R \circ \psi_0^{0,\infty} \circ R^{-1}, \tag{2.33}$$

where $R(w) = w^N$. Hence $\tilde{\psi}_0^{0,\infty}(w) = w(\xi_0^{0,\infty}(w))^n$.

PROOF. We have that $f^q = g^{Nq}$. Let $\tilde{\Phi}_j^{\pm}$ be the Fatou coordinates of g^q. Then the Fatou coordinates $\Phi_j^{\pm}$ of f are given by $\Phi_j^{\pm} = \frac{1}{N}\tilde{\Phi}_j^{\pm} = L_{1/N} \circ \tilde{\Phi}^{\pm}$, where $L_{\alpha}(W) = \alpha W$. Hence

$$\tilde{\Psi}_j^0 = \tilde{\Phi}_j^{-} \circ (\tilde{\Phi}_j^{+})^{-1} = L_N \circ \Phi_j^{-} \circ (\Phi_j^{+})^{-1} \circ L_{1/N}, \tag{2.34}$$

and similarly for $\tilde{\Psi}_j^{\infty}$. The rest of the proof follows as in Corollary 2.8. □

It is the conditions (2.27) and (2.32) which we will explain in Section 4 below.

There are other kinds of root extraction problems. Although we expect similar explanations we will not consider them here, except for one, since the corresponding map f is not of codimension 1. In all cases there can be formal obstructions due to improper multiplicity, as described above for the root extraction of f tangent to the identity when $g'(0) \neq 1$ (see Lemma 2.5). The one exception we analyze now is the case of a map

$$f(z) = z + z^{q+1} + o(z^{q+1}) \tag{2.35}$$

and the existence of a q-th root of the form $g(z) = \exp(2\pi i/q)z + o(z)$. The Ecalle-Voronin modulus of (2.35) is a $2q$-tuple of germs of analytic functions $(\psi_0^0, \psi_0^\infty, \dots, \psi_{q-1}^0, \psi_{q-1}^\infty)$ and the condition for the existence of g is

$$\psi_{j+1}^{0,\infty} = B^{-1} \circ \psi_j^{0,\infty} \circ B \tag{2.36}$$

with B given in (2.11). We explain below in Theorem 4.3 the meaning of this condition.

2.4 The Lavaurs maps and the renormalized return maps

When $\epsilon \neq 0$ there are global maps $L_{j,\epsilon} : \mathbb{S}_{j+1,\epsilon}^- \to \mathbb{S}_{j,\epsilon}^+$ identifying points with the same orbits (see Figure 1), called the Lavaurs maps. As these maps are global analytic diffeomorphisms of the sphere preserving 0 and ∞ they are linear. With the choice of coordinates yielding (2.8) all $L_{j,\epsilon}$ are identical: we simply call them L_ϵ. The exact expression for L_ϵ can be calculated explicitly under (2.8) but the result depends whether $\epsilon \in V_+$ or $\epsilon \in V_-$, where $V_\pm$ are given in (2.14) ([14]). We do not give the exact value since it is not needed here. The Lavaurs maps allow to define the renormalized return maps on $\mathbb{S}_{j,\epsilon}^+$ in the neighborhood of 0 or ∞ by composing the $\psi_{j,\epsilon}^{0,\infty}$ with L_ϵ. A renormalized return map is just a first return map for the orbit of a point on the crescent $S_{j,\epsilon}^+$, but written in the spherical coordinate on $\mathbb{S}_{j,\epsilon}^+$ ($\mathbb{S}_{j,\epsilon}$ is the quotient of $S_{j,\epsilon}^+$ under f_ϵ^q). It is defined as follows: taking a point z in the crescent $S_{j,\epsilon}^+$ we follow its orbit forward under f_ϵ^q until we come back for the first time to the crescent $S_{j,\epsilon}^+$. The corresponding point is $\kappa(z) = f_\epsilon^{Mq}(z)$ for some M. Such a map κ is defined in the neighborhood of each end of the crescent. Its expression in the spherical coordinate on $\mathbb{S}_{j,\epsilon}^+$ is the renormalized return map. While M may not depend continuously on z the expression of the renormalized return map is analytic in the spherical coordinate, including at 0 or ∞. Depending on ϵ, the points 0 and ∞ represent either the fixed point 0 and one of the periodic points z_j of f, or the converse (see Figure 1).

Multipliers at the fixed points of the renormalized return maps. While the Lavaurs maps depend on the parametrization of the spheres the multipliers of the renormalized return maps are intrinsic. If $S_{j,\epsilon}^+$ is bounded by curves ℓ and $f_\epsilon^q(\ell)$ crossing at z_m, where $(f_\epsilon^q)'(z_m) = \exp(2\pi i\alpha)$, then the renormalized return map at the corresponding point of $\mathbb{S}_{j,\epsilon}^+$ has multiplier $\exp(-\frac{2\pi i}{\alpha})$ (see for instance [15] or [8]).

3 The renormalized return map and parametric resurgence phenomenon

The renormalized return maps are only really interesting in the Siegel directions.

The renormalized return maps of f_ϵ^q are defined in the neighborhoods of the representatives of the fixed points z_j of f_ϵ^q on the spheres $\mathbb{S}_\ell^\pm$. When $q > 1$, the fixed points of f_ϵ^q are of two types: $z_0 = 0$ is the fixed point of f_ϵ, while $z_1, \dots, z_q$ are periodic points of f_ϵ of period q. For $q = 1$, f_ϵ has two fixed points $\pm\sqrt{\epsilon}$ and there are renormalized return maps in the neighborhood of each of them.

The renormalized return maps are defined as germs of diffeomorphisms $(\mathbb{S}^+_{j,\epsilon}, 0) \to (\mathbb{S}^+_{j,\epsilon}, 0)$ and $(\mathbb{S}^+_{j,\epsilon}, \infty) \to (\mathbb{S}^+_{j,\epsilon}, \infty)$ and identify points belonging to the same orbits of f_ϵ^q.

As the two discussions are completely identical we will limit ourselves to discuss the case $(\mathbb{S}^+_{1,\epsilon}, 0) \to (\mathbb{S}^+_{1,\epsilon}, 0)$.

Theorem 3.1 *(1) We consider a germ of generic family f_ϵ in prepared form*

$$f_\epsilon(z) = z + (z^2 - \epsilon)(1 + A(\epsilon) + (z^2 - \epsilon)h(z, \epsilon)). \tag{3.1}$$

Let us suppose that for some value ϵ_0 the renormalized return map for f_{ϵ_0} at $-\sqrt{\epsilon_0}$ (resp. $\sqrt{\epsilon_0}$) is resonant of order 1 (the first coefficient of the normal form is nonzero). Then f_{ϵ_0} is resonant at $-\sqrt{\epsilon_0}$ (resp. $\sqrt{\epsilon_0}$) of order 1.

(2) We consider a germ of generic family f_ϵ in prepared form unfolding a resonant diffeomorphism with multiplier $\exp(2\pi i \frac{p}{q})$ and its q-th iterate

$$f_\epsilon^q(z) = z + z(z^q - \epsilon)(1 + A(\epsilon) + (z^q - \epsilon)h(z, \epsilon)). \tag{3.2}$$

Let us suppose that for some value ϵ_0 the renormalized return map for $f_{\epsilon_0}^q$ at the origin (resp. at one of the periodic points z_j of f_{ϵ_0}) is resonant of order 1. Then f_{ϵ_0} is resonant at the origin (resp. at z_j) of order 1.

Proof. We only prove (2) as (1) is similar and a bit simpler. In (2), there are four cases to consider, depending whether ϵ is in the negative or positive Siegel direction of the origin and of the periodic orbit. We discuss two of these as the two others are completely similar.

Case 1: The first case is when ϵ_0 is in the negative Siegel direction for the periodic orbit and we consider for instance the renormalized return map at z_1 which, in the spherical coordinate w on $\mathbb{S}^+_{1,\epsilon}$ (see Figure 1), is the germ of map k_{1,ϵ_0} at 0. We drop the first index and simply write k_{ϵ_0} and k_ϵ for its unfolding. The map has the form $k_{\epsilon_0}(w) = L_{\epsilon_0} \circ \psi^0_{0,\epsilon}(w) = \exp(2\pi i r/m)w + o(w) = \exp(2\pi i r/m - 2\pi^2 a/q)\psi^0_{1,\epsilon_0}(w)$. This means that

$$(f_{\epsilon_0}^q)'(z_1) = \exp\left(-\frac{2\pi i m}{r + mn}\right) \tag{3.3}$$

for some $n \in \mathbb{N}$ (see for instance [8] or [14]). The only periodic orbits which can bifurcate from the orbit $(z_1, \dots, z_q)$ for $\epsilon = \epsilon_0$ have period $q(r + mn)$ ($r + mn$ points bifurcate from each z_j). The lifting $K_\epsilon(W)$ of k_ϵ to the Z-plane has the form

$$K_\epsilon = \Phi^+_{1,\epsilon} \circ T_{-q\alpha} \circ (\Phi^+_{1,\epsilon})^{-1}, \tag{3.4}$$

(see [14]) where in general T_β is the translation

$$T_\beta(Z) = Z + \beta \tag{3.5}$$

and

$$\alpha(\epsilon) = \begin{cases} \frac{2\pi i}{q\epsilon} & \epsilon \neq 0 \\ 0 & \epsilon = 0. \end{cases} \tag{3.6}$$

We are interested in a neighborhood of ϵ_0. Since $k'_{\epsilon_0}(0) = \exp(2\pi i r/m)$ with $(r,m)=1$, we have

$$\lim_{Im(W)\to-\infty} K_{\epsilon_0}^m(W) = W + M, \tag{3.7}$$

with $M \in \mathbb{Z}$, while there exists $R>0$ such that

$$K_{\epsilon_0}^m(W) \neq W + M \qquad \forall W \quad \text{such that} \quad Im(W) < -R. \tag{3.8}$$

Moreover

$$\lim_{Im(W)\to-\infty} K_{\epsilon_0}^d(W) - W \notin \mathbb{Z} \tag{3.9}$$

if d is a strict divisor of m.

Let $W = \Phi_{1,\epsilon_0}^+(Z)$. Then $K_{\epsilon_0}(\Phi_{1,\epsilon_0}^+(Z)) = \Phi_{1,\epsilon_0}^+ \circ T_{-q\alpha_0}(Z)$. From (3.7)

$$\lim_{Im(Z)\to-\infty} K_{\epsilon_0}^m(\Phi_{1,\epsilon_0}^+(Z)) = \Phi_{1,\epsilon_0}^+(Z) + M. \tag{3.10}$$

Suppose now that $k_{\epsilon_0}^m(w) = w + bw^{m+1} + o(w^{m+1})$ with $b \neq 0$. Hence, for ϵ close to ϵ_0, the map k_ϵ has a unique periodic orbit $(w_1(\epsilon),\dots,w_m(\epsilon))$ such that $w_j(\epsilon_0)=0$. This yields points $W_j(\epsilon)$, $j=1,\dots,m$, such that $K_\epsilon^m(W_j(\epsilon)) = W_j(\epsilon) + M(\epsilon)$ with $M(\epsilon)\in\mathbb{Z}$ and such that $\lim_{\epsilon\to\epsilon_0} Im(W_j(\epsilon)) = -\infty$. Moreover if $Z_j(\epsilon)$ are such that $\Phi_{1,\epsilon}^+(Z_j(\epsilon)) = W_j(\epsilon)$, then

$$K_\epsilon^m(\Phi_{1,\epsilon}^+(Z_j(\epsilon))) = \Phi_{1,\epsilon}^+ \circ T_{-qm\alpha}(Z_j(\epsilon)) = \Phi_{1,\epsilon}^+(Z_j(\epsilon)) + M(\epsilon) = \Phi_{1,\epsilon}^+(F_\epsilon^{qM(\epsilon)}(Z_j(\epsilon))). \tag{3.11}$$

Because of (3.7) we get $M(\epsilon)\equiv M$. Moreover as $\Phi_{1,\epsilon}^+$ is a diffeomorphism we get

$$F_\epsilon^{qM}(Z_j(\epsilon)) = T_{-qm\alpha}(Z_j(\epsilon)). \tag{3.12}$$

Hence if $y_j(\epsilon) = p_\epsilon(Z_j(\epsilon))$, then $f_\epsilon^{qM}(y_j(\epsilon)) = y_j(\epsilon)$, i.e. $y_j(\epsilon)$ is a periodic point of f_ϵ^q whose period divides M. Let us show that the period is exactly M. Indeed suppose that the period is $d|M$. Then $M = dc$. From (3.12) we need have $c|m$, i.e. $m = cd'$. Then $F_\epsilon^{qd}(Z_j(\epsilon)) = T_{-qd'\alpha}(Z_0)$, which implies $\Phi_{1,\epsilon}(F_\epsilon^{qd}(Z_j(\epsilon))) = K_\epsilon^{d'}(\Phi_{1,\epsilon}(Z_j(\epsilon)))+d$, a contradiction with (3.9).

Moreover $\lim_{\epsilon\to\epsilon_0} y_j(\epsilon) = z_1$. From (3.3) we have that $M = r+mn$. Note that we have obtained only m points of a periodic orbit of f_ϵ of period $q(mn+r)$: it comes from the fact that we have only obtained the points of the orbit which belong to the crescent $S_{1,\epsilon}^+$ associated to the sphere $\mathbb{S}_{1,\epsilon}^+$. The remaining points of the orbit are obtained by taking the iterates of the $y_j(\epsilon)$ under f_ϵ.

We now look at the normal form of f_0^q. It can either be linear or have nonzero resonant terms. We start by the latter and suppose that the normal form for $f_{\epsilon_0}^q$ be given by $f_{\epsilon_0}^q(z) = \exp(-2\pi i\frac{m}{r+mn})(z-z_1)+c(z-z_1)^{\ell(r+mn)+1}+o(z^{\ell(r+mn)+1})$ with $c\neq 0$ and $\ell>1$. As $\frac{\partial^2 f_\epsilon}{\partial z\partial\epsilon}\neq 0$ in a neighborhood of $(0,0)$ in (z,ϵ) space, then the small perturbation f_ϵ^q of $f_{\epsilon_0}^q$ would have ℓ periodic orbits of period $r+mn$ bifurcating from z_1 (counting multiplicities). These would produce ℓ small orbits of periodic points of period m of k_ϵ for ϵ close to ϵ_0, a contradiction.

If the normal form of $f_{\epsilon_0}^q$ has no nonzero resonant terms then f_{ϵ_0} is analytically linearizable, yielding that $f_{\epsilon_0}^{q(r+mn)}$ is the identity. This implies that $F_{\epsilon_0}^{qM} = T_{-qm\alpha}$. It then follows that

$$\Phi_{1,\epsilon}^+ \circ T_{-qm\alpha} = T_M \circ \Phi_{1,\epsilon}^+. \tag{3.13}$$

Hence

$$T_M = \Phi_{1,\epsilon}^+ \circ T_{-qm\alpha} \circ (\Phi_{1,\epsilon}^+)^{-1} = K_\epsilon^m. \tag{3.14}$$

The last equality follows from (3.4), yielding that K_ϵ^m is a translation, a contradiction with (3.8).

Case 2: The second case is the case when ϵ_0 is in the positive Siegel direction for the origin and we consider values ϵ_0 for which $f'_{\epsilon_0}(0) = \exp\left(2\pi i\left(\frac{p}{q} - \frac{m}{q(r+mn)}\right)\right)$. Let k_{ϵ_0} be the renormalized return map at the origin which is defined on $\mathbb{S}_{1,\epsilon}^+$. As this case is very similar to the previous case we only write the differences. The map has the form $k_{\epsilon_0}(w) = \exp(2\pi i r/m)(w) + o(w)$.

From Figure 1, we see that $k_{\epsilon_0}(w) = L \circ \psi_{1,\epsilon_0}^0 \circ L \circ \cdots \circ L \circ \psi_{q-1,\epsilon_0}^0 \circ L \circ \psi_{0,\epsilon_0}^0(w)$. The lifting $K_\epsilon(W)$ of k_ϵ to the Z-plane has the form $K_\epsilon = \Phi_{1,\epsilon}^+ \circ T_{-q^2\alpha} \circ (\Phi_{1,\epsilon}^+)^{-1}$ (see [14]).

We are interested in a neighborhood of ϵ_0. We have

$$\lim_{Im(W)\to-\infty} K_{\epsilon_0}^m(W) = W + M \in \mathbb{Z}. \tag{3.15}$$

Let $W = \Phi_{1,\epsilon_0}^+(Z)$. Then $K_{\epsilon_0}(\Phi_{1,\epsilon_0}^+(Z)) = \Phi_{1,\epsilon_0}^+ \circ T_{-q^2\alpha_0}(Z)$. Hence $\lim_{Im(Z)\to-\infty} K_{\epsilon_0}^m(\Phi_{1,\epsilon_0}^+(Z)) = \Phi_{1,\epsilon_0}^+(Z) + M$.

Suppose now that $k_{\epsilon_0}^m(w) = w + bw^{m+1} + o(w^{m+1})$ with $b \neq 0$. Hence for ϵ close to ϵ_0 the map k_ϵ has a periodic orbit $(w_1(\epsilon), \dots, w_m(\epsilon))$ such that $w_j(\epsilon_0) = 0$. This yields points $W_j(\epsilon)$ such that $K_\epsilon^m(W_j(\epsilon)) = W_j(\epsilon) + M$ and $\lim_{\epsilon\to\epsilon_0} Im(W_j(\epsilon)) = -\infty$. Moreover if $Z_j(\epsilon)$ are such that $\Phi_{1,\epsilon}^+(Z_j(\epsilon)) = W_j(\epsilon)$, then

$$K^m(\Phi_{1,\epsilon}^+(Z_j(\epsilon))) = \Phi_{1,\epsilon}^+ \circ T_{-q^2m\alpha}(Z_j(\epsilon)) = \Phi_{1,\epsilon}^+(Z_j(\epsilon)) + M = \Phi_{1,\epsilon}^+(F_\epsilon^{qM}(Z_j(\epsilon))). \tag{3.16}$$

We get $F_\epsilon^{qM}(Z_j(\epsilon)) = T_{-q^2m\alpha}(Z_j(\epsilon))$. Hence if $y_j(\epsilon) = p_\epsilon(Z_j(\epsilon))$, then $f_\epsilon^{qM}(y_j(\epsilon)) = y_j(\epsilon)$, i.e. $y_j(\epsilon)$ is a periodic point of f_ϵ^q whose period divides M. Let us show that the period is exactly M. Indeed suppose that the period is $d|M$. Then $M = dc$ and $c|m$, so that $m = cd'$. Hence $F_\epsilon^{qd}(Z_j(\epsilon)) = T_{-q^2d'\alpha}(Z_0)$, which implies $\Phi_{1,\epsilon}(F_\epsilon^{qd}(Z_j(\epsilon))) = K_\epsilon^{d'}(\Phi_{1,\epsilon}(Z_j(\epsilon))) + d$, a contradiction with the minimality of m satisfying (3.15). As before we conclude that $M = r + mn$.

Moreover $\lim_{\epsilon\to\epsilon_0} y_j(\epsilon) = 0$.

Now suppose that the normal form for f_{ϵ_0} be given by $f_{\epsilon_0}(z) = \exp(2\pi i(\frac{p}{q} - \frac{m}{q(r+mn)}))z + cz^{\ell q(r+mn)+1} + o(z^{\ell q(r+mn)+1})$, with $c \neq 0$ and $\ell > 1$. Then as before, for ϵ close to ϵ_0, f_ϵ would have ℓ periodic orbits of period Mq bifurcating from the origin (counting multiplicities). These would produce ℓ small orbits of periodic points of period m of k_ϵ for ϵ close to ϵ_0, a contradiction. Hence $\ell = 1$. The case of a linear normal form also yields a contradiction as in Case 1.

Cases 3 and 4: The last two cases are similar, the only difference being that we work in regions which are neighborhoods of $Im(W) = +\infty$. □

4 The root extraction problem

We now explain the condition (2.27) of Theorem 2.7.

Theorem 4.1 *We consider a generic germ of analytic diffeomorphism f as in* (2.25)*.*

(1) If f has an N-th root g tangent to the identity, then there exists a generic family f_ϵ unfolding f (i.e. $f_0 = f$ for $\epsilon = 0$) such that, for all sufficiently small ϵ, f_ϵ has an N-th root.

(2) If f_ϵ has an N-th root g_ϵ which is tangent to the identity for $\epsilon = 0$, then the components $\psi_{j,\epsilon}^{0,\infty}$ of its modulus all satisfy

$$\psi_{j,\epsilon}^{0,\infty}(w) = w\xi_{j,\epsilon}^{0,\infty}(w^N). \tag{4.1}$$

Then the modulus of the N-th root g_ϵ has the form $(\tilde{\psi}_\epsilon^0, \tilde{\psi}_\epsilon^\infty)$ with

$$\tilde{\psi}_\epsilon^{0,\infty}(w) = R \circ \psi_\epsilon^{0,\infty} \circ R^{-1}, \tag{4.2}$$

where $R(w) = w^N$.

(3) If f has no N-th root tangent to the identity, then for any generic family f_ϵ unfolding f, i.e. such that $\frac{\partial f_\epsilon}{\partial \epsilon} \neq 0$, (hence f_ϵ has two distinct fixed points z_1 and z_2 for $\epsilon \neq 0$), then there exists a neighborhood U of the origin such that, for sufficiently small ϵ, f_ϵ never has an N-th root over U.

The obstruction materializes in particular in the following way: there exists $\ell \in \{1, 2\}$ and a sequence (ϵ_n) converging to the origin, so that the germ of f_{ϵ_n} at z_ℓ has no N-th root. For each n there exists a small neighborhood V_n of ϵ_n, a small neighborhood U_n of z_ℓ and an integer $M(n)$ such that, for $\epsilon \in V_n$, f_ϵ has a unique periodic orbit of period $M(n)$ in U_n, which is an obstruction to finding an N-th root of f_ϵ over U_n. The periodic orbit coallesces with z_ℓ as $\epsilon = \epsilon_n$. Moreover $M(n) \uparrow +\infty$ as $n \to \infty$.

Meaning of part (3) of Theorem 4.1: as it is the important part of our paper it is worth taking some time to discuss what is the meaning of the conclusion. If f has no N-root, then either ψ^0 or ψ^∞ has a nonzero monomial whose exponent is not of the form $kN + 1$. Let us take the lowest such monomial and call it our "obstruction". The map ψ^0 (resp. ψ^∞) controls the dynamics near $-\sqrt{\epsilon}$ (resp. $\sqrt{\epsilon}$) (see Figure 3). We localize the diffeomorphism f_ϵ at $-\sqrt{\epsilon}$ (resp. $\sqrt{\epsilon}$) if the obstruction is a monomial of ψ^0 (resp. ψ^∞). To study the dynamics of f_ϵ near this point we rather study the dynamics of its renormalized return map as in Section 3. We focus on special values of ϵ, namely the sequence ϵ_n, where the renormalized return map has a fixed multiplier. The multiplier is chosen so that the "obstruction" becomes a resonant monomial of the renormalized return map of first order. Then, when we perturb ϵ_n, we get a unique periodic orbit which is an obstruction to taking an N-th root. In the case of the multiplier being equal to 1, this is the phenomenon described in Lemma 2.5.

Proof of Theorem 4.1.

(1) Let f have an N-th root g tangent to the identity, and let g_ϵ be a generic family (i.e. $\frac{\partial g_\epsilon}{\partial \epsilon} \neq 0$) unfolding g. Then $f_\epsilon = g_\epsilon^N$ is a family unfolding f and having an N-th root.

(2) Let us derive the condition for f_ϵ to have an N-th root in terms of the Fatou coordinates: it is the same as for the case $\epsilon = 0$. The Fatou coordinates Φ for f_ϵ satisfy

$$\Phi(F_\epsilon(Z)) = \Phi(Z) + 1. \tag{4.3}$$

Let us suppose that $f_\epsilon = g_\epsilon^N$ with g_ϵ analytic and g_0 tangent to the identity. Let G_ϵ be its lifting in the Z-coordinate. Let Φ_1 be a Fatou coordinate for g_ϵ. Then

$$\Phi_1(G_\epsilon(Z)) = \Phi_1(Z) + 1, \tag{4.4}$$

yielding $\Phi_1(F_\epsilon(Z)) = \Phi_1(Z) + N$. Then $\Phi(Z) = \frac{1}{N}\Phi_1(Z)$ is a Fatou coordinate for F_ϵ. From (4.4) we get

$$G_\epsilon(Z) = \Phi_1^{-1} \circ T_1 \circ \Phi_1 = \Phi^{-1} \circ T_{\frac{1}{N}} \circ \Phi. \tag{4.5}$$

In order that g_ϵ exists we need that G_ϵ commutes with T_α, i.e.

$$T_\alpha \circ \Phi^{-1} \circ T_{\frac{1}{N}} \circ \Phi = \Phi^{-1} \circ T_{\frac{1}{N}} \circ \Phi \circ T_\alpha. \tag{4.6}$$

We can rewrite this

$$(\Phi \circ T_{-\alpha} \circ \Phi^{-1}) \circ T_{\frac{1}{N}} = T_{\frac{1}{N}} \circ (\Phi \circ T_{-\alpha} \circ \Phi^{-1}) \tag{4.7}$$

i.e. the map $T_{\frac{1}{N}}$ commutes with the renormalized return map $K = \Phi \circ T_{-\alpha} \circ \Phi^{-1}$. The renormalized return map is the composition of the Lavaurs translation with a map Ψ (Ψ is either Ψ_ϵ^0 or Ψ_ϵ^∞ satisfying $\Psi(W+1) = \Psi(W) + 1$). As $\Psi(W) = W + \sum_{n\in\mathbb{Z}} b_n \exp(2\pi i n W)$ we need that $b_n = 0$ as soon as N does not divide n. At the level of the modulus ($\psi_\epsilon^{0,\infty}$) the condition is

$$\psi_\epsilon^{0,\infty}(w) = \exp\left(-\frac{2\pi i}{N}\right) \psi_\epsilon^{0,\infty}\left(\exp\left(\frac{2\pi i}{N}\right) w\right). \tag{4.8}$$

The last part is as in Corollary 2.8.

(3) The first part comes from the fact that it is possible to cover a neighborhood of the origin in ϵ-space with two sectors and on each sector to define $\psi_\epsilon^{0,\infty}$ depending continuously on ϵ near $\epsilon = 0$.

We can always suppose (modulo a change of coordinate and parameter) that the family f_ϵ is "prepared", i.e. the fixed points are located at $\pm\sqrt{\epsilon}$ and let us suppose that $z_\ell = -\sqrt{\epsilon}$. Then

$$\psi_0^0(w) = c_1 w + \sum_{j=1}^{r} c_j w^{Nj+1} + C w^{m+1} + o(w^{m+1}) \tag{4.9}$$

with $c_1, C \neq 0$ and $Nr < m < N(r+1)$. The sequence ϵ_n is chosen such that the first return map $k_{\epsilon_n}^0 = L \circ \psi_{\epsilon_n}^0$ has a multiplier $\exp(2\pi i/m)$, where L is the Lavaurs map and $f'_{\epsilon_n}(0) = \exp(-2\pi i \frac{m}{1+mn})$. As $\psi_\epsilon^0(w) = \psi_0^0(w) + O(\epsilon)$, then, for n sufficiently large the normal form of $k_{\epsilon_n}^0$ is of the form

$$\exp(2\pi i/m) w + C'(\epsilon) w^{m+1} + o(w^{m+1}), \tag{4.10}$$

with $C'(\epsilon) \neq 0$. Indeed, when we remove the terms in w^{jN+1}, this only creates higher order terms of the same form, so the process can never annihilate the term in w^{m+1}. Hence at $\epsilon = \epsilon_n$ we have the birth of a unique periodic orbit of k_ϵ of period m. Then the lifting $K_{\epsilon_n}^m$ is such that

$$\lim_{Im(W)\to-\infty} K_{\epsilon_n}^m = W + M(n) \tag{4.11}$$

for some $M(n) \in \mathbb{Z}$. For ϵ close to ϵ_n this yields to the birth of a periodic orbit of f_ϵ of period $M(n) = 1 + mn$ for ϵ close to ϵ_n (see Theorem 3.1). Let us now suppose that $f_\epsilon = g_\epsilon^N$ with $g'_{\epsilon_n}(-\sqrt{\epsilon}) = \exp(-2\pi i \frac{m}{N(1+mn)})$. Then the periodic orbit of period $M(n)$ yields a periodic orbit of period exactly $NM(n)$ for g_ϵ, since $N \nmid m$ and the orbit coallesces with z_ℓ for $\epsilon \to \epsilon_n$. This in turn yields N periodic orbits of period $M(n)$ for f_ϵ, a contradiction. □

Similarly we explain the condition (2.27) of Theorem 2.7.

Theorem 4.2 *We consider a generic germ of analytic diffeomorphism f as in* (2.31)*.*

(1) If f has an N-th root then there exists a generic family f_ϵ unfolding f such that, for all sufficiently small ϵ, then f_ϵ has an N-th root.

(2) If f_ϵ has an N-th root then the components $\psi_{j,\epsilon}^{0,\infty}$ of its modulus satisfy

$$\psi_{j,\epsilon}^{0,\infty}(w) = w\xi_{j,\epsilon}^{0,\infty}(w^N). \tag{4.12}$$

(3) If f has no N-root then, for any generic family f_ϵ unfolding f, i.e. such that $\frac{\partial^2 f_\epsilon}{\partial z \partial \epsilon} \neq 0$, then there exists a neighborhood U of the origin such that, for sufficiently small ϵ, f_ϵ never has an N-th root over U.

The obstruction materializes in particular in the following way:

(i) there exists a sequence (ϵ_n) converging to the origin, so that the germ of f_{ϵ_n} at the origin has no N-th root. For each n there exists a small neighborhood V_n of ϵ_n, a small neighborhood U_n of the origin and an integer $M(n)$ such that, for $\epsilon \in V_n$, f_ϵ has a unique periodic orbit of period $M(n)$ in U_n which is an obstruction to finding an N-th root of f_ϵ over U_n. The periodic orbit coallesces with the origin as $\epsilon = \epsilon_n$. Moreover $M(n) \uparrow +\infty$ as $n \to \infty$;

(ii) there exists a second sequence (ϵ'_n) so that the germ of $f_{\epsilon'_n}^q$ localized at a fixed point z_j (a periodic point of $f_{\epsilon'_n}$) has no N-th root. For each n there exists a small neighborhood V'_n of ϵ'_n, a small neighborhood U'_n of z_j and an integer $M'(n)$ such that, for $\epsilon \in V'_n$, f_ϵ^q has a unique periodic orbit of period $M'(n)$ in U'_n which is an obstruction to finding an N-th root of f_ϵ over U'_n. The periodic orbit coallesces with the fixed point z_j of $f_{\epsilon_n}^q$ as $\epsilon = \epsilon'_n$. Moreover $M'(n) \uparrow +\infty$ as $n \to \infty$.

Proof. The proofs of (1) and (2) are similar to the ones in Theorem 4.1. We now consider (3). Then one of ψ_1^0 (resp. ψ_1^∞) does not satisfy (2.32). If it is ψ_1^0 (resp. ψ_1^∞) then the sequence of values ϵ_n will be chosen in the Siegel domain in the direction in which ψ_1^0 (resp.

ψ_1^∞) controls the dynamics of the origin and the sequence of ϵ_n' also in the Siegel domain, in the direction in which ψ_1^0 (resp. ψ_1^∞) controls the dynamics of the periodic orbit.

Let us treat the case of ψ_1^0. For the sequence (ϵ_n') we consider the renormalized return map for $f_{\epsilon_n'}^q$ which is given by $k_{\epsilon_n'} = L \circ \psi_{1,\epsilon_n'}^0$ for an appropriate L so that all $k_{\epsilon_n'}$ be resonant of order 1 with same multiplier $\exp(2\pi i/m)$. Then as in Theorem 3.1 the ϵ_n' are such that $(f_{\epsilon_n'}^q)'(z_1) = \exp\left(-\frac{2\pi i m}{1+nm}\right)$. The rest of the proof is as in Theorem 4.1.

The case of the sequence ϵ_n requires a little more work. Let m be defined as in (4.9). The relation between the $\psi_{j,\epsilon}^0$ is given in (2.9). Let s be such that $sp \equiv -1 \pmod q$. Then $\psi_{j-1,\epsilon}^0 = B^{-s} \circ \psi_{j,\epsilon}^0 \circ B^s$. The renormalized return map k_{ϵ_n} of $f_{\epsilon_n}^q$ in the neighborhood of 0 is given by:

$$\begin{aligned} k_{\epsilon_n} &= L \circ \psi_{1,\epsilon_n}^0 \circ L \circ \dots \circ \psi_{q-1,\epsilon_n}^0 \circ L \circ \psi_{0,\epsilon_n}^0 \\ &= L \circ (B^{-(q-1)s} \circ \psi_{0,\epsilon_n}^0 \circ B^{(q-1)s}) \circ L \circ \dots \circ (B^{-s} \circ \psi_{0,\epsilon_n}^0 \circ B^s) \circ L \circ \psi_{0,\epsilon_n}^0 \\ &= B^{-(q-1)s} \circ L \circ \psi_{0,\epsilon_n}^0 \circ L_1 \circ \dots \circ \psi_{0,\epsilon_n}^0 \circ L_1 \circ \psi_{0,\epsilon_n}^0 \\ &= (L_1 \circ \psi_{0,\epsilon_n}^0)^q \end{aligned} \tag{4.13}$$

where $L_1 = L \circ B^s$, since linear maps commute and $B^{sq} = id$. The sequence (ϵ_n) is chosen so that $f_{\epsilon_n}'(0) = \exp\left(2\pi i \left(\frac{p}{q} - \frac{m}{q(1+mn)}\right)\right)$, which implies that the multiplier of $L_1 \circ \psi_0^0$ at the origin is given by $\exp(\frac{2\pi i}{mq})$. Then it is easily checked that k_{ϵ_n} also has the form (4.9) (since composition of maps of this type yields a map of this type) with multiplier at the origin $\exp(\frac{2\pi i}{m})$. The rest of the proof is as in Theorem 4.1. □

Theorem 4.3 *We consider a germ of analytic diffeomorphism*

$$f(z) = z + z^{q+1} + o(z^{q+1}). \tag{4.14}$$

(1) If f has a q-th root g with $g'(0) = \exp(2\pi i/q)$, then there exists a generic family f_ϵ unfolding f such that, for all sufficiently small ϵ, f_ϵ has a q-th root.

(2) If f has no q-th root g, with $g'(0) = \exp(2\pi i/q)$, then for any generic one-parameter "prepared" family f_ϵ unfolding f, i.e. f_ϵ is of the form

$$f_\epsilon(z) = z + z(z^q - \epsilon)(1 + A(\epsilon) + (z^q - \epsilon)h(z, \epsilon)) \tag{4.15}$$

and satisfies the properties described in Section 2.1, there exists a neighborhood U of the origin such that, for sufficiently small ϵ, f_ϵ never has a q-th root over U. Also:

(i) There exists a sequence (ϵ_n) converging to the origin, so that the germs of f_{ϵ_n} at z_j have no q-th root. For each n and each z_j there exists a small neighborhood V_n of ϵ_n, a small neighborhood U_n of z_j and an integer $M(n)$ such that, for $\epsilon \in V_n$, f_ϵ has a unique periodic orbit of period $M(n)$ in U_n which is an obstruction to finding a q-th root of f_ϵ over U_n. The periodic orbit coallesces with z_j as $\epsilon = \epsilon_n$. Moreover $M(n) \uparrow +\infty$ as $n \to \infty$.

(ii) There exists a sequence (ϵ_n') converging to the origin, so that the germ of $f_{\epsilon_n'}$ at the origin has no q-th root. For each n there exists a small neighborhood V_n' of ϵ_n', a small neighborhood U_n' of the origin and an integer $M'(n)$ such that, for $\epsilon \in V_n'$, f_ϵ has a unique periodic orbit of period $M'(n)$ in U_n' which is an obstruction to finding a q-th root of f_ϵ over U_n'. Moreover $M'(n) \uparrow +\infty$ as $n \to \infty$.

Proof. We only discuss (2). As f has no q-th root then there exists j such that $\psi_{j+1}^0 \neq B^{-1}\circ\psi_j^0\circ B$ or $\psi_{j+1}^\infty \neq B^{-1}\circ\psi_j^\infty\circ B$. We only discuss the first case. Let $m+1$ be the first order where this is not true. Depending if ϵ is in the positive or negative Siegel direction we will consider the renormalized return map $k_{j,\epsilon}$ near z_j (case (i)) or the renormalized return map $k_{0,\epsilon}$ at the origin (case(ii)).

(i) Because of the hypothesis there exists at least one j such that $(\psi_{j,0}^0)^{(m+1)}(0)\neq 0$. We consider the renormalized return map $k_{j,\epsilon}$ near z_j and we choose the sequence ϵ_n so that $k'_{j,\epsilon_n}(0) = (\psi_{j,\epsilon_n}^0)'(0)L'_{\epsilon_n}(0) = \exp(\frac{2\pi i}{m})$ (z_j is represented by 0 on $\mathbb{S}_{j,\epsilon}^+$). This yields, for ϵ close to ϵ_n, the birth of a periodic orbit $\{w_1,\dots w_m\}$ of period m for $k_{j,\epsilon}$. We suppose that the germ of f_ϵ at z_j has a square root g_ϵ for ϵ close to ϵ_n. The points w_ℓ belong to $\mathbb{S}_{j,\epsilon}^+$. Their images $\tilde{w}_\ell\in\mathbb{S}_{j+1,\epsilon}^+$ under the dynamics of g_ϵ are given by $\tilde{w}_\ell = B^{-1}(w_\ell)$ must be periodic points of $k_{j+1,\epsilon}$ of period m. The points w_ℓ (resp. $\tilde{w}_\ell$) are fixed points of $k_{j,\epsilon}^m$ (resp. $k_{j+1,\epsilon}^m$). Because $\psi_{j+1}^0\neq B^{-1}\circ\psi_j^0\circ B$ at the order $m+1$ the equation $B^{-1}\circ k_{j,\epsilon}^m(w) - k_{j+1,\epsilon}^m\circ B^{-1}(w) = 0$ has exactly $m+1$ small zeros which should be 0 and the w_ℓ. But 0 is a double root of this equation, yielding a contradiction. An obstruction at the level of periodic points of the renormalized return maps obviously yields one at the level of periodic points of f_ϵ. We do not discuss the details.

(ii) The renormalized return map $k_{0,\epsilon}$ near 0 is given by

$$k_{0,\epsilon} = L\circ\psi_{1,\epsilon}^0\circ\dots\circ L\circ\psi_{q-1,\epsilon}^0\circ L\circ\psi_{0,\epsilon}^0 \tag{4.16}$$

We let $\chi_{j,\epsilon}^0 = B^j\circ\psi_{j,\epsilon}^0\circ B^{-j}$ (in particular $\chi_{0,\epsilon}^0 = \psi_{0,\epsilon}^0$). Note that B and L commute and $B^q = id$ and let $M = L\circ B^{-1}$.

$$\begin{aligned} k_{0,\epsilon} &= L\circ(B^{-1}\circ\chi_{1,\epsilon}^0\circ B)\circ\dots\circ L\circ(B^{-(q-1)}\circ\chi_{q-1,\epsilon}^0\circ B^{q-1})\circ L\circ\psi_{0,\epsilon}^0 \\ &= M\circ\chi_{1,\epsilon}^0\circ\dots\circ M\circ\chi_{q-1,\epsilon}^0\circ M\circ\chi_{0,\epsilon}^0. \end{aligned} \tag{4.17}$$

The hypothesis is that

$$\chi_{j,\epsilon} = \sum_{\ell=1}^m c_\ell(\epsilon)z^\ell + C_j(\epsilon)z^{m+1} + o(z^{m+1}), \tag{4.18}$$

where the $c_\ell(0)$ are independent of j and there exists j and j' such that $C_j\neq C_{j'}$. We take ϵ'_n such that $(M\circ\chi_{0,\epsilon'_n}^0)'(0) = \exp(2\pi i\frac{s}{mq})$, for some $s\in\{1,\dots,m-1\}$ to be chosen below. Let h be a map which brings $M\circ\chi_{0,\epsilon'_n}^0$ to normal form up to order $m+1$:

$$h^{-1}\circ M\circ\chi_{0,\epsilon'_n}^0\circ h = \exp\left(2\pi i\frac{s}{mq}\right)w + D_0w^{m+1}. \tag{4.19}$$

We apply the change of coordinate to $M\circ\chi_{j,\epsilon'_n}^0$ and get

$$h^{-1}\circ M\circ\chi_{j,\epsilon'_n}^0\circ h = \exp\left(2\pi i\frac{s}{mq}\right)w + D_{q-j}w^{m+1}. \tag{4.20}$$

Moreover necessarily $D_j\neq D_{j'}$ for small ϵ'_n. Let $\tau = \exp(\frac{2\pi i}{qm})$. Then

$$h^{-1}\circ k_{0,\epsilon'_n}\circ h(w) = \tau^{sq}(w) + \tau^{s(q-1)}\left(\sum_{j=0}^{q-1} D_j\tau^{sjm}\right)w^{m+1} + o(w^{m+1}). \tag{4.21}$$

The system

$$\sum_{j=0}^{q-1} D_j \tau^{sjm} = 0, \qquad s = 1, \dots q-1 \tag{4.22}$$

has a matrix of rank $q-1$, since it contains a Vandermonde submatrix. Hence the set of solutions has dimension 1. As $\sum_{j=0}^{q-1} \tau^{sjm} = 0$ they are all multiple of $(1, 1, \dots, 1)$. So there exists one s such that $\sum_{j=0}^{q-1} D_j \tau^{sjm} \neq 0$.

This yields that

$$k_{\epsilon_n'}^m = w + C(\epsilon_n')w^{m+1} + o(w^{m+1}) \tag{4.23}$$

with $C(\epsilon_n') \neq 0$. Hence for ϵ close to ϵ_n' we have the birth of a unique periodic orbit of period m for $k_{0,\epsilon}$. As before this orbit is an obstruction to the existence of a q-th root of f_ϵ. $\square$

5 The problem of conformal equivalence of curvilinear angles in conformal geometry

Conformal geometry is the study of properties of geometric configurations which are invariant under all conformal transformations. We limit ourselves to germs of regular real analytic arcs in the plane and regular conformal transformations in a region including the arcs. If the region is identified to an open set of $\mathbb{C}$ then a regular conformal transformation is identified to a holomorphic diffeomorphism on that region. Any germ of single curve can be transformed into the germ of the real axis at the origin. So the first non trivial configuration is composed of two germs of curves having a common point, i.e. a curvilinear angle. We will suppose that the common point is the origin. The particular case where the two curves are straight lines will be called the linear angle. The problem of conformal equivalence of two such curvilinear angles has been studied by Kasner ([5], [6]), Pfeiffer ([10], [11]) and, more recently, by Nakai [9] and Ahern & Gong [1] (none of these authors have considered the unfoldings.)

Definition 5.1 A curvilinear angle, (C_1, C_2) formed by two germs of real analytic curves C_1 and C_2 intersecting at the origin is conformally equivalent to a second curvilinear angle (C_1', C_2') if there exists a germ of holomorphic diffeomorphism h at the origin such that $h(C_i) = C_i'$, $i = 1, 2$.

Obviously the angle θ between the two curves is a conformal invariant. Kasner has proved that there are no other formal conformal invariant if $\beta = \theta/\pi \notin \mathbb{Q}$ and that there exists a formal change of coordinate

$$z \mapsto \sum_{n=1}^{\infty} a_n z^n \tag{5.1}$$

sending the curvilinear angle to the linear angle. However divergence is the rule and convergence the exception when $\beta = \theta/\pi$ is Liouvillian. Here we will discuss the case $\theta/\pi \in \mathbb{Q}$. In the generic case there is a formal obstruction to bring the curvilinear angle to the linear angle and one gets as formal invariants an integer k (to be thought of as the codimension) and one real number a [6]. This yields for each odd k and a a unique "model". For an even k and given a there can be two models [1]: this comes from the fact that the transformation sending one to the other does not preserve the "real" character of the problem. In all cases we have generic divergence of a transformation (5.1) sending a curvilinear angle to the model curvilinear angle. The equivalence classes of curvilinear angles with same invariants k and a have been first studied by Nakai [9]. Ahern & Gong [1] completed the construction of a complete modulus of conformal classification.

Here we will explain the meaning of the invariants and of the modulus by studying a deformation of the curvilinear angle.

We consider two germs of regular analytic curves C_1 and C_2 such that the curves C_1 and C_2 cut at an angle $\theta = \pi\frac{p}{q}$ at the origin. We also consider the case of a zero angle corresponding to $p = 0$, i.e. the two curves are tangent, which is called a *horn*. We can of course suppose that C_1 is the real axis. For each curve we consider the germ of Schwarz reflection Σ_j with respect to the curve C_j. Then $\Sigma_1(z) = \overline{z}$ and $C_j = \mathrm{Fix}(\Sigma_j)$. We consider the map:

$$f = \Sigma_2 \circ \Sigma_1. \tag{5.2}$$

Then f is a germ of resonant analytic diffeomorphism with

$$f(z) = \exp\left(2\pi i\frac{p}{q}\right) z + o(z). \tag{5.3}$$

Moreover from the definition of f and the fact that the Schwarz reflections are involutions we have that

$$\Sigma_1 \circ f = f^{-1} \circ \Sigma_1. \tag{5.4}$$

Definition 5.2 The diffeomorphism $f = \Sigma_2 \circ \Sigma_1$ is called the *diffeomorphism associated to the curvilinear angle* (C_1, C_2).

Lemma 5.3 *We consider two curvilinear angles (C_1, C_2) and (C_1', C_2') and let f and f' be their respective associated diffeomorphisms. For each curve C_j (resp. C_j') we consider the germ of Schwarz reflection Σ_j (resp. Σ_j') with respect to the curve. Then $C_j = Fix(\Sigma_j)$ and $C_j' = Fix(\Sigma_j')$. The two curvilinear angles (C_1, C_2) and (C_1', C_2') are conformally equivalent under the conformal equivalence h if and only if*

$$h \circ \Sigma_j = \Sigma_j' \circ h, \tag{5.5}$$

which yields $h \circ f = f' \circ h$, i.e. h conjugates f and f'.

Conversely given two germs of analytic diffeomorphisms f and f' and two germs of Schwarz reflections Σ_1 and Σ_1' such that $\Sigma_1 \circ f = f^{-1} \circ \Sigma_1$ and $\Sigma_1' \circ f' = (f')^{-1} \circ \Sigma_1'$ which are conjugate under h satisfying $h \circ \Sigma_1 = \Sigma_1' \circ h$, then h is a conjugacy between the curvilinear angles (C_1, C_2) and (C_1', C_2'), where

$$\begin{cases} \Sigma_2 = f \circ \Sigma_1 \\ \Sigma_2' = f' \circ \Sigma_1' \\ C_j = Fix(\Sigma_j) \\ C_j' = Fix(\Sigma_j'). \end{cases} \tag{5.6}$$

Proof. If h is a conformal equivalence between (C_1, C_2) and (C_1', C_2') then (5.5) is satisfied. Also

$$h \circ f = h \circ \Sigma_2 \circ \Sigma_1 = \Sigma_2' \circ h \circ \Sigma_1 = \Sigma_2' \circ \Sigma_1' \circ h = f' \circ h. \tag{5.7}$$

Conversely we only need to show that $h \circ \Sigma_2 = \Sigma_2' \circ h$. This follows from

$$h \circ \Sigma_2 = h \circ (\Sigma_2 \circ \Sigma_1) \circ \Sigma_1 = h \circ f \circ \Sigma_1 = f' \circ h \circ \Sigma_1 = f' \circ \Sigma_1' \circ h = \Sigma_2' \circ h. \tag{5.8}$$

□

When we consider the problem of conformal equivalence of curvilinear angles (C_1, C_2) and (C_1', C_2'), we can of course suppose that we have applied conformal transformations sending C_1 and C_1' to the real axis. Then the problem of conformal equivalence between the two angles is equivalent to the problem of conjugacy of the associated diffeomorphisms under a conjugacy h satisfying $h \circ \Sigma = \Sigma \circ h$ where

$$\Sigma(z) = \overline{z}. \tag{5.9}$$

5.1 The modulus of conformal classification

We limit our discussion to the generic case where, up to a change of coordinates, the map f_0 can be written in one of the forms

$$f_0(z) = z + iz^2 + o(z^2) \tag{5.10}$$

and

$$f_0(z) = \exp\left(2\pi i \frac{p}{q}\right) z + Cz^{q+1} + o(z^{q+1}), \tag{5.11}$$

with $\arg(C) = -2\pi \frac{p}{q} \pm \frac{\pi}{2}$. In the latter case the only linear changes of coordinates which commute with Σ are the changes $z \mapsto cz$ with $c \in \mathbb{R}$. Depending whether q is even or odd, we can bring f_0 to:

$$f_0(z) = \begin{cases} \exp(2\pi i \frac{p}{q}) z + \frac{1}{q} \exp(-2\pi i \frac{p}{q} + \frac{\pi i}{2}) z^{q+1} + o(z^{q+1}) & q \text{ even} \\ \exp(2\pi i \frac{p}{q}) z + \frac{1}{q} \exp(-2\pi i \frac{p}{q} \pm \frac{\pi i}{2}) z^{q+1} + o(z^{q+1}) & q \text{ odd.} \end{cases} \tag{5.12}$$

In the case of (5.10) the two curves have a contact of order 2. If we unfold the curves in families $C_{j,\epsilon}$ of curves depending analytically on ϵ, then the two unfolded curves $C_{j,\epsilon}$ can have two intersection points or none. Note however that the unfolded map f_ϵ always has two fixed points. The two fixed points are the intersection points of $C_{1,\epsilon}$ and $C_{2,\epsilon}$ when the two curves intersect. They are outside the curves when the two curves do not intersect, but they control the geometry if we want to describe the conformal geometry over a fixed neighborhood of the origin throughout the perturbation (Figure 4). If we call $P_{1,\epsilon}$ and $P_{2,\epsilon}$ the two fixed points of f_ϵ, then, for ϵ in the Poincaré region, we have that $\Sigma_{j,\epsilon}(P_{1,\epsilon}) = P_{2,\epsilon}$, $j = 1, 2$.

The presence of $P_{1,\epsilon}$ and $P_{2,\epsilon}$ when ϵ is in the Poincaré region puts a limit on the size of a neighborhood on which we can send two non intersecting arcs to two non intersecting arcs. In general the neighborhood must not contain both $P_{1,\epsilon}$ and $P_{2,\epsilon}$.

In the case of (5.11), one easily sees that the obstruction to linearize the family (i.e. to bring the curvilinear angle to the linear angle) comes from the fact that, in all unfoldings, f_ϵ has a unique small periodic orbit of period q. The uniqueness comes from the fact that the diffeomorphism f_0 is generic (the codimension k is equal to 1).

Because of the fact that we consider unfoldings f_ϵ satisfying

$$f_\epsilon \circ \Sigma = \Sigma \circ f_\epsilon^{-1} \tag{5.13}$$

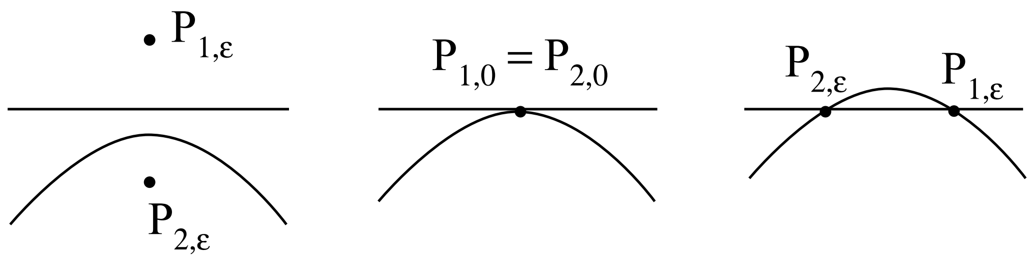


Figure 4: The points $P_{1,\epsilon}$ and $P_{2,\epsilon}$

we need to adapt the definition of "prepared family". Indeed we need to compare f_ϵ with a model family which satisfies (5.13). For a generic family f_ϵ unfolding (5.12) we prefer to give a model family for its q-th iterate: such a model family is given by the time-one map of

$$i\frac{z(z^q-\epsilon)}{1+A(\epsilon)z^q}\,\frac{\partial}{\partial z}. \tag{5.14}$$

For a family unfolding (5.10) it is given by the time-one map of

$$i\frac{z^2-\epsilon}{1+A(\epsilon)z}\,\frac{\partial}{\partial z}. \tag{5.15}$$

In both cases we limit ourselves to real values of ϵ and to real $A(\epsilon)$. A family f_ϵ is prepared if the fixed points z_j of f_ϵ^q coincide with the singular points of the vector field and if the multipliers λ_j of f_ϵ^q at the fixed points z_j of f_ϵ^q are of the form $\lambda_j = \exp(\mu_j)$ where μ_j is the eigenvalue of the singular point z_j of the vector field. The fixed points z_j are either real or come in pairs $z_j, \overline{z}_j$. The $\mu_j = \mu(z_j)$ satisfy $\mu(z_j) + \mu(\overline{z}_j) = 0$, so $\mu(z_j) \in i\mathbb{R}$ when $z_j \in \mathbb{R}$. Equivalently $\lambda(z_j)\lambda(\overline{z}_j) = 1$. As the formal invariant is given by $a(\epsilon) = \sum \frac{1}{\mu_j}$, then $a(\epsilon) = -iA(\epsilon) \in i\mathbb{R}$. From this the following proposition follows easily.

Proposition 5.4 *We consider a generic family f_ϵ unfolding* (5.12) *and satisfying* (5.13) *where Σ is given in* (5.9). *The only admissible values of ϵ are such that $|f_\epsilon'(0)| = 1$ and $|(f_\epsilon^q)'(z_j)| = 1$, where $z_1, \dots, z_q$ are the periodic points of f_ϵ of period q. It is possible to find a change of coordinate and parameter $(z,\epsilon) \mapsto (\tilde{z}, \tilde{\epsilon})$ so as to bring the family to a prepared family $\tilde{f}_{\tilde{\epsilon}}$ which still satisfies* (5.13).

Proof. We give few details as we do not want to recall the full details of how the preparation is done in [14]. If a family is in normal form up to order q then the equation for singular points has the form

$$(1+O(\epsilon))z^q - \epsilon + o(z^q) = 0. \tag{5.16}$$

which we bring to $\hat{z}^q - \hat{\epsilon} = 0$ by means of a change of coordinate $\hat{z} = z(1+O(\epsilon)) + o(z)$. We then make a scaling in $\hat{z}$ and a change of parameter to bring the family to the prepared form. As the initial system satisfies (5.13) (so the singular points either satisfy $\Sigma(z_j) = z_j$ or come in symmetric pairs) and the condition to fulfill in the prepared form also satisfies (5.13) it is easily to see that the change $(z,\epsilon) \mapsto (\tilde{z}, \tilde{\epsilon})$ preserves (5.13). □

The unfolding of (5.10) also yields the geometric interpretation of the quantity $A(\epsilon)$:

Proposition 5.5 *When the two curves intersect at $\pm\sqrt{\epsilon}$ in the unfolding of a curvilinear angle, the angles at the intersection points are given by*

$$\theta_{\pm} = \frac{\pm\sqrt{\epsilon}}{1 \pm A(\epsilon)\sqrt{\epsilon}}. \tag{5.17}$$

In particular when $A(0) \neq 0$ the angles can never be opposite in the unfolding. $A(\epsilon)$ yields a measure of the difference between the two angles through the relation:

$$\frac{1}{\theta_+} + \frac{1}{\theta_-} = A(\epsilon). \tag{5.18}$$

For the next theorem we use the Martinet-Ramis point of view for the modulus of analytic classification of a resonant diffeomorphism. The case $\epsilon = 0$ was done in [9] and [1].

Theorem 5.6 *We consider a generic prepared family f_ϵ unfolding (5.12) satisfying (5.13). Then for admissible values of ϵ the formal invariant $a(\epsilon)$ is pure imaginary. For an adequate choice of coordinates on the spheres, the modulus of orbital analytic classification given by the 2-tuple $(\tilde{\psi}_\epsilon^0, \tilde{\psi}_\epsilon^\infty)$ (in the Martinet-Ramis point of view) satisfies for admissible values of ϵ*

$$\begin{cases} \tilde{\psi}_\epsilon^0 = \Sigma \circ (\tilde{\psi}_\epsilon^0)^{-1} \circ \Sigma \\ \tilde{\psi}_\epsilon^\infty = \Sigma \circ (\tilde{\psi}_\epsilon^\infty)^{-1} \circ \Sigma. \end{cases} \tag{5.19}$$

Proof. Because of the condition (5.13) we need to take the conjugate $\tilde{f}_\epsilon^q = r \circ f_\epsilon^q \circ r^{-1}$ of f_ϵ^q by the rotation $r(z) = \tilde{z} = \exp(\frac{\pi i}{2q})z$ in order to apply directly the results of [14]. Let $s = r \circ \Sigma \circ r^{-1}$. Then $s \circ \tilde{f}_\epsilon^q = \tilde{f}_\epsilon^{-q} \circ s$. We have $s(z) = \exp(\frac{\pi i}{q})\overline{z}$: it is the symmetry with respect to the line making an angle $\frac{\pi}{2q}$ with the horizontal axis. Geometrically this means that, when embedding $\tilde{f}_\epsilon^q$ in a flow, the trajectories look symmetric with respect to this line as in Figure 1.

In the original coordinate the map looks as in Figure 5 with a horizontal symmetry axis.

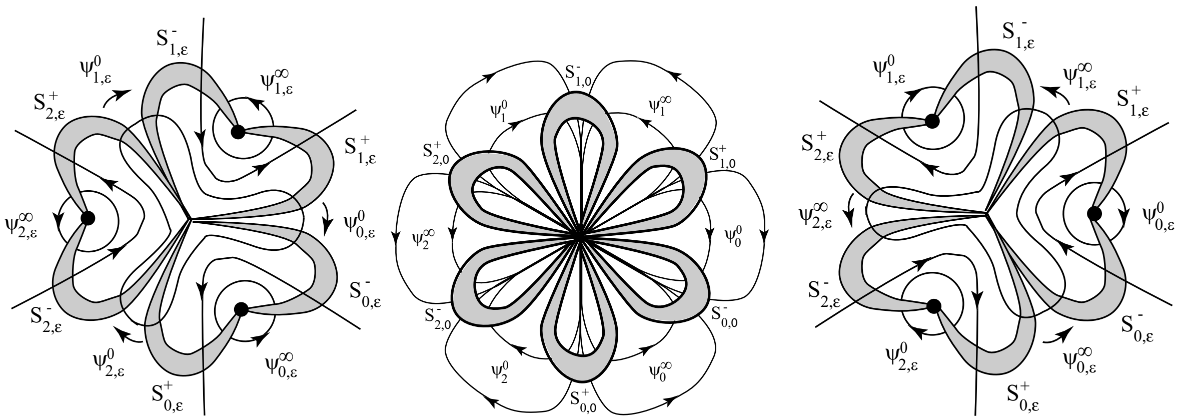
To construct the Fatou coordinates we make the change of parameter $\eta = i\epsilon$ and of variable

$$Z = p_\eta^{-1}(\tilde{z}) = \begin{cases} \frac{1}{q\eta} \ln \frac{\tilde{z}^q - \eta}{\tilde{z}} & \eta \neq 0 \\ -\frac{1}{q\tilde{z}^q} & \eta = 0. \end{cases} \tag{5.20}$$

We let F_η^q be the map $\tilde{f}_\eta^q$ in Z-coordinate. As $\eta \in i\mathbb{R}$ this yields

$$S \circ F_\eta^q = F_\eta^{-q} \circ S, \tag{5.21}$$

where the map S is an involution on the Z-space (see Figure 2) defined as follows: S is the identity on the half-line $l = \{Z | ReZ = 0, ImZ < 0\}$ located in the lower part of the translation domain Q_0^- (which we can view as the image of the positive real axis in z-space). Let Z_0 belong to this half-line and let $\alpha(t)$ and $\beta(t)$, $t \in [0,1]$ be curves in Z-space with $\alpha(0) = \beta(0) = Z_0$ whose projections on $\tilde{z}$-space (resp. z-space) are symmetric under s (resp. Σ). We define $S(\alpha(1)) = \beta(1)$. S is well defined and is a kind of generalized symmetry with respect to the vertical direction on the q-sheeted Z-space. Moreover S is an involution:



(a) Negative Siegel direction (b) $\epsilon = 0$ (c) Positive Siegel direction

Figure 5: The maps $\psi_{j,\epsilon}$ for ϵ in the Siegel direction

$S^{-1} = S$. In the lower part of the translation domains Q_0^- and Q_1^+ it looks like $Z \mapsto -\overline{Z}$. Elsewhere there is an additional change of sheet.

Once we have identified the symmetry (5.21) on F_η^q we can follow what it means for the modulus. We choose $\Phi_{1,\eta}^+$ and $\Phi_{0,\eta}^-$ so that

$$S \circ \Phi_{0,\eta}^- = \Phi_{1,\eta}^+ \circ S, \tag{5.22}$$

where by abuse of notation we also define $S(W) = -\overline{W}$. We still have freedom for one base point Z_0 such that $\Phi_{1,\eta}^+(Z_0) = 0$. The base point is chosen so that the set of Fatou coordinates is normalized (i.e. (2.8) is valid). This means in particular that the translation terms in $\Psi_{j,\eta}^{0,\infty}$ are real.

All the other Fatou coordinates are then determined by the rule

$$\Phi_{\sigma(j),\eta}^\pm \circ F_\eta = T_{1/q} \circ \Phi_{j,\eta}^\pm \tag{5.23}$$

As $\Psi_{0,\eta}^0 = \Phi_{0,\eta}^- \circ (\Phi_{1,\eta}^+)^{-1}$ this yields

$$S \circ \Psi_{0,\eta}^0 = (\Psi_{0,\eta}^0)^{-1} \circ S \tag{5.24}$$

Let us now consider $\tilde{\Psi}_\eta^\infty$. Let m be such that $mp \equiv 1 \pmod q$. Then from (2.17)

$$\begin{cases} \Phi_{1,\eta}^- = \Phi_{\sigma^m(0),\eta}^- = T_{\frac{m}{q}} \circ \Phi_{0,\eta}^- \circ F_\eta^{-m} \\ \Phi_{0,\eta}^+ = \Phi_{\sigma^{-m}(1),\eta}^+ = T_{-\frac{m}{q}} \circ \Phi_{1,\eta}^+ \circ F_\eta^m. \end{cases} \tag{5.25}$$

Hence, as $\Psi_{j,\eta}^\infty = \Phi_{j,\eta}^- \circ (\Phi_{j,\eta}^+)^{-1}$,

$$\Psi_{0,\eta}^\infty = \Phi_{0,\eta}^- \circ (\Phi_{0,\eta}^+)^{-1} = \Phi_{0,\eta}^- \circ F_\eta^{-m} \circ (\Phi_{1,\eta}^+)^{-1} \circ T_{\frac{m}{q}}, \tag{5.26}$$

and

$$\begin{aligned}
\Psi_{1,\eta}^{\infty} &= \Phi_{1,\eta}^{-}\circ(\Phi_{1,\eta}^{+})^{-1}\\
&= T_{\frac{m}{q}}\circ\Phi_{0,\eta}^{-}\circ F_{\eta}^{-m}\circ(\Phi_{1,\eta}^{+})^{-1}\\
&= T_{\frac{m}{q}}\circ S\circ\Phi_{1,\eta}^{+}\circ S\circ F_{\eta}^{-m}\circ S\circ(\Phi_{0,\eta}^{-})^{-1}\circ S\\
&= T_{\frac{m}{q}}\circ S\circ\Phi_{1,\eta}^{+}\circ F_{\eta}^{m}\circ(\Phi_{0,\eta}^{-})^{-1}\circ S\\
&= S\circ T_{-\frac{m}{q}}\circ\Phi_{1,\eta}^{+}\circ F_{\eta}^{m}\circ(\Phi_{0,\eta}^{-})^{-1}\circ S\\
&= S\circ(\Psi_{0,\eta}^{\infty})^{-1}\circ S
\end{aligned} \tag{5.27}$$

At the level of the small $\psi_{j,\eta}^{0,\infty}=E\circ\Psi_{j,\eta}^{0,\infty}\circ E^{-1}$ with $E(W)=\exp(-2\pi iW)$ this yields:

$$\begin{cases}\psi_{0,\eta}^{0}(w)=\Sigma\circ(\psi_{0,\eta}^{0})^{-1}\circ\Sigma\\ \psi_{0,\eta}^{\infty}(w)=\Sigma\circ(\psi_{1,\eta}^{\infty})^{-1}\circ\Sigma\end{cases} \tag{5.28}$$

with $\Sigma(w)=E\circ S\circ E^{-1}(w)=\overline{w}$. Hence, using (2.22) we have

$$\tilde{\psi}_{\eta}^{\infty}=\ell\circ\psi_{1,\eta}^{\infty}=\ell\circ\Sigma\circ(\psi_{0,\eta}^{\infty})^{-1}\circ\Sigma=\Sigma\circ\ell^{-1}\circ(\psi_{0,\eta}^{\infty})^{-1}\circ\Sigma=\Sigma\circ(\tilde{\psi}_{\eta}^{\infty})^{-1}\circ\Sigma, \tag{5.29}$$

yielding (5.19). □

Corollary 5.7 *We consider a generic family f_ϵ unfolding* (5.12) *and satisfying* (5.13)*. Then for admissible $\epsilon\in i\mathbb{R}$ it is possible to construct Fatou coordinates satisfying*

$$S\circ\Phi_j^{-}=\Phi_{q+1-j}^{+}\circ S. \tag{5.30}$$

(Indices are $(\mathrm{mod}\, q)$.)

Theorem 5.8 *([1] for the case $\epsilon=0$) We consider two prepared families f_ϵ and $\tilde{f}_\epsilon$ unfolding f_0 and $\tilde{f}_0$, with same multiplier* $\exp(2\pi i\frac{p}{q})$*, of the same type* (5.12)*, and both satisfying* (5.13)*. Then they are conjugate under a conjugacy h_ϵ commuting with Σ if and only if they have the same formal invariant a and the same modulus $(\psi_{1,\epsilon}^{0},\psi_{1,\epsilon}^{\infty})$.*

Proof. It is already known from [14] that the two families are conjugate. We only need to prove that there exists a conjugacy between the associated diffeomorphisms which commutes with Σ. Let $\Phi_{j,\epsilon}^{\pm}$ (resp. $\tilde{\Phi}_{j,\epsilon}^{\pm}$) be the Fatou coordinates of f_ϵ (resp. $\tilde{f}_\epsilon$). As the conjugacy is obtained by the compositions

$$p_\epsilon\circ(\tilde{\Phi}_{j,\epsilon}^{\pm})^{-1}\circ\Phi_{j,\epsilon}^{\pm}\circ(p_\epsilon)^{-1} \tag{5.31}$$

which glue together as a global map this follows from Corollary 5.7 and the fact that $\Sigma\circ p_\epsilon = p_\epsilon\circ S$. □

Theorem 5.9 *We consider a generic prepared family f_ϵ unfolding* (5.10) *and satisfying* (5.13)*. Then, for admissible values of $\epsilon\in\mathbb{R}$, the formal invariant $a(\epsilon)$ is pure imaginary. For an adequate choice of coordinate on the spheres, the modulus of orbital analytic classification given by $(\psi_\epsilon^0,\psi_\epsilon^\infty)$ satisfies for admissible values of ϵ*

$$\begin{cases}\psi_{\epsilon}^{0}=\Sigma\circ(\psi_{\epsilon}^{0})^{-1}\circ\Sigma\\ \psi_{\epsilon}^{\infty}=\Sigma\circ(\psi_{\epsilon}^{\infty})^{-1}\circ\Sigma.\end{cases} \tag{5.32}$$

Proof. The proof is identical to that of Theorem 5.9. □

5.2 The problem of conformal bisection of a generic curvilinear rational angle

Here we consider the problem of conformal bisection of a generic rational angle. A curvilinear zero angle is called a "horn" in the terminology of [5].

Definition 5.10 (1) A curvilinear angle (C_1, C_2) can be bisected if there exists a germ of analytic curve C_3 such that $\Sigma_3(C_1) = C_2$, where Σ_3 is the Schwarz reflection associated to C_3. If the diffeomorphism f associated with (C_1, C_2) has multiplier $\exp(2\pi i\frac{p}{q})$, then C_3 is an internal (resp. external) bisector if the diffeomorphism associated to (C_1, C_3) has multiplier $\exp(\pi i\frac{p}{q})$ (resp. $\exp(\pi i\frac{p+q}{q})$).

(2) A curvilinear angle (C_1, C_2) can be sected in N equal parts if there exists $N-1$ curves C_j, $j = 3, \dots, N+1$, and their associated Schwarz reflections Σ_j such that

$$\begin{gathered}\Sigma_3(C_1) = C_4\\ \Sigma_4(C_3) = C_5\\ \vdots\\ \Sigma_{N+1}(C_N) = C_2\end{gathered} \tag{5.33}$$

Proposition 5.11 *A curvilinear angle can be sected in N equal parts if and only if its associated diffeomorphism has an N-th root which satisfies* (5.13).

Proof. If the angle can be sected in N equal parts then

$$f = \Sigma_2 \circ \Sigma_1 = (\Sigma_2 \circ \Sigma_{N+1}) \circ (\Sigma_{N+1} \circ \Sigma_N) \cdots \circ (\Sigma_3 \circ \Sigma_1). \tag{5.34}$$

Moreover (5.33) implies that

$$\begin{cases}\Sigma_4 = \Sigma_3 \circ \Sigma_1 \circ \Sigma_3,\\ \Sigma_5 = \Sigma_4 \circ \Sigma_3 \circ \Sigma_4,\\ \qquad\vdots\\ \Sigma_2 = \Sigma_{N+1} \circ \Sigma_N \circ \Sigma_{N+1}\end{cases} \tag{5.35}$$

which yields

$$\Sigma_3 \circ \Sigma_1 = \Sigma_4 \circ \Sigma_3 = \cdots = \Sigma_2 \circ \Sigma_{N+1}. \tag{5.36}$$

Hence $f = g^N$ with $g = \Sigma_3 \circ \Sigma_1$. Moreover $g \circ \Sigma_j = \Sigma_j \circ g^{-1}$ for all j.

Conversely, let us suppose that $f = g^N$ for some g satisfying $g \circ \Sigma_1 = \Sigma_1 \circ g^{-1}$. Let $\Sigma_3 = g \circ \Sigma_1$, $\Sigma_4 = g \circ \Sigma_3$, …, $\Sigma_{N+1} = g \circ \Sigma_N$ and $C_j = \text{Fix}(\Sigma_j)$. Then the C_j satisfy (5.33). □

As we have seen in Section 2.3 (for instance Lemma 2.5) the two problems of internal and external bisection of the angle can be very different: in some cases there are formal obstructions which can be seen at the level of a finite jet, in addition to the analytic obstructions which exist in all cases and can never be seen at the level of a finite germ.

We discuss the different cases: the first case will be discussed at length and the others more briefly.

(i) Internal section of the horn into $N \geq 2$ parts.

This case is covered by Theorem 4.1. Indeed if f_0 satisfies (5.13) and has an N-th root, then an N-th root can be found which also satisfies (5.13). If f_0 has no N-th root, then the Siegel direction is the direction in which the two curves $C_{j,\epsilon}$ intersect. We consider the renormalized return map of f_{ϵ_n} at one of the intersection points, for instance $-\sqrt{\epsilon_n}$ and let us take the case where the renormalized return map is calculated with $\psi^0_{\epsilon_n}$. The values ϵ_n are chosen so that the curves intersect with an angle $\frac{m\pi}{1+mn}$, where m is as in (4.9). The corresponding $f_{\epsilon_n}^{1+mn}$ has a fixed point of multiplicity $2+mn$. Let us now consider values of ϵ close to ϵ_n. Then, multiplying the angle by $1+mn$, (the inverse of secting the angle in $1+mn$ equal parts) yields to curves C_{1,ϵ_n} and C'_{2,ϵ_n} intersecting with an angle $m\pi$ (see Figures 6-8). The two curves $C_{1,\epsilon}$ and $C'_{2,\epsilon}$ intersect in one, two or three points in the 1-parameter family for ϵ close to ϵ_n: one is the intersection point $-\sqrt{\epsilon}$ of $C_{1,\epsilon}$ and $C_{2,\epsilon}$, while the others are periodic points of f_ϵ of an orbit of exact period $1+mn$. The exact number of intersection points of $C_{1,\epsilon}$ and $C'_{2,\epsilon}$ depends on the parity of $1+mn$ and on ϵ (see Figures 6-8).

So the interesting question is to give the meaning of the other periodic points of f_ϵ. For that purpose we embed our curves $C_{j,\epsilon}$, $j=1,2$, in a sequence of curves $(\tilde{C}_{j,\epsilon})|_{j\in\mathbb{Z}}$. The curves $\tilde{C}_{j,\epsilon}$ are obtained by taking copies of the angle. They are given by $\tilde{C}_{\ell,\epsilon} = \mathrm{Fix}(\tilde{\Sigma}_{\ell,\epsilon})$ where

$$\tilde{\Sigma}_{\ell,\epsilon} = f_\epsilon^{\ell-1} \circ \Sigma_1, \qquad \ell \in \mathbb{Z}. \tag{5.37}$$

From this definition we have

$$\begin{cases} \tilde{C}_{1,\epsilon} = C_{1,\epsilon} \\ \tilde{C}_{2,\epsilon} = C_{2,\epsilon} = \mathrm{Fix}(\Sigma_{2,\epsilon}) \\ \tilde{C}_{3,\epsilon} = \mathrm{Fix}(\Sigma_{2,\epsilon}\circ\Sigma_1\circ\Sigma_{2,\epsilon}) \\ \tilde{C}_{4,\epsilon} = \mathrm{Fix}(\Sigma_{2,\epsilon}\circ\Sigma_1\circ\Sigma_{2,\epsilon}\circ\Sigma_1\circ\Sigma_{2,\epsilon}) \\ \vdots \\ \tilde{C}_{0,\epsilon} = \mathrm{Fix}(\Sigma_1\circ\Sigma_{2,\epsilon}\circ\Sigma_1) \\ \tilde{C}_{-1,\epsilon} = \mathrm{Fix}(\Sigma_1\circ\Sigma_{2,\epsilon}\circ\Sigma_1\circ\Sigma_{2,\epsilon}\circ\Sigma_1) \\ \vdots \end{cases} \tag{5.38}$$

In particular $\tilde{C}_{j+1,\epsilon}$ is the bisector of $\tilde{C}_{j,\epsilon}$ and $\tilde{C}_{j+2,\epsilon}$. We now prove that the periodic points represent points where the curves $\tilde{C}_{j,\epsilon}$ are likely to intersect.

Theorem 5.12 *We consider the family of curves $\tilde{C}_{\ell,\epsilon}$ given by $\tilde{C}_{\ell,\epsilon} = Fix(\tilde{\Sigma}_{\ell,\epsilon})$ where $\tilde{\Sigma}_{\ell,\epsilon}$ is defined in* (5.37)*. If f_ϵ has a unique orbit $\{z_1,\dots,z_N\}$ of period N, then necessarily some periodic point z_s of the orbit lies either on $C_{1,\epsilon}$ or on $C_{2,\epsilon}$. In the first case, if $\Sigma_1(z_s)=z_s$ then, for all $\ell\in\mathbb{Z}$, $z_{\ell+s}\in Fix(\tilde{\Sigma}_{2\ell+1,\epsilon})$, where the index for z is modulo N. In the second case $\Sigma_{2,\epsilon}(z_s)=z_s$ then, for all $\ell\in\mathbb{Z}$, $z_{\ell+s}\in Fix(\tilde{\Sigma}_{2\ell+2,\epsilon})$.*

Proof. Let $\{z_1,\dots,z_N\}$ be the unique orbit of period N and let us suppose that $z_j = f_\epsilon^{j-1}(z_1)$. If z_j is a periodic point of period N, then $\overline{z}_j = \Sigma_1(z_j)$ and $\Sigma_{2,\epsilon}(z_j)$ are also periodic points of period N. Let us apply this to z_1. As we have a unique orbit of period N, then

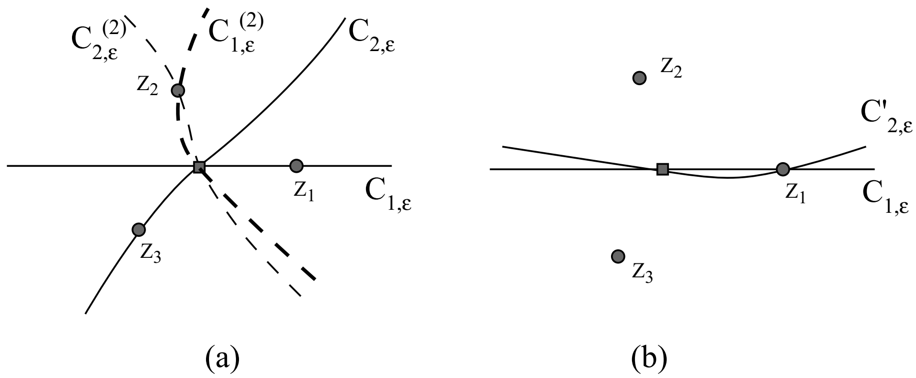


Figure 6: $C_{1,\epsilon}$, $C_{2,\epsilon}$, $\tilde{C}_{0,\epsilon}$, $\tilde{C}_{3,\epsilon}$ and $C'_{2,\epsilon}$ for $m=1$, $n=2$ and ϵ near ϵ_2

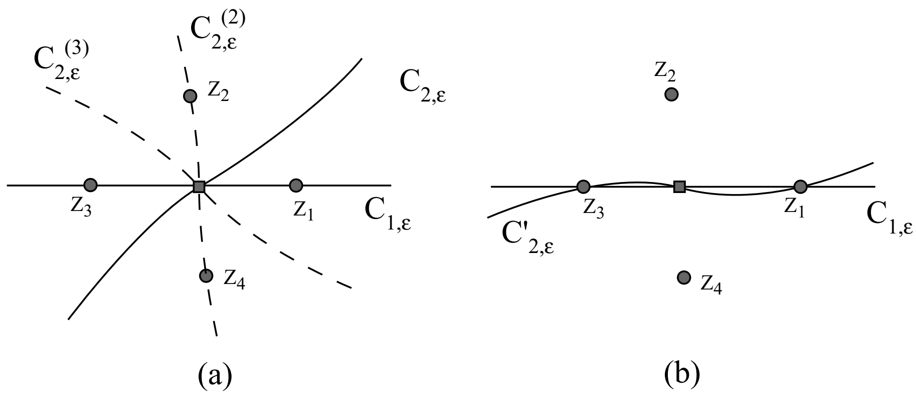


Figure 7: $C_{1,\epsilon}$, $C_{2,\epsilon}$, $\tilde{C}_{3,\epsilon}$, $\tilde{C}_{4,\epsilon}$ and $C'_{2,\epsilon}$ for $m=1$, $n=3$ and ϵ near ϵ_3

there exists j such that $\overline{z_1} = \Sigma_1(z_1) = z_j = f_\epsilon^{j-1}(z_1)$. We will distinguish the cases j odd and j even. If $j = 2\ell+1$, then

$$f_\epsilon^{-\ell} \circ \Sigma_1(z_1) = \Sigma_1 \circ f_\epsilon^\ell(z_1) = f_\epsilon^\ell(z_1), \tag{5.39}$$

i.e. $\Sigma_1(z_{\ell+1}) = z_{\ell+1}$. If $j = 2\ell$, then

$$\begin{array}{rcl} f_\epsilon^{-\ell} \circ \Sigma_{2,\epsilon}(z_1) & = & f_\epsilon^{-\ell+1} \circ (\Sigma_1 \circ \Sigma_{2,\epsilon}) \circ \Sigma_{2,\epsilon}(z_1) \\ & = & f_\epsilon^{-\ell+1} \circ \Sigma_{1,\epsilon}(z_1) \\ & = & f_\epsilon^\ell(z_1). \end{array} \tag{5.40}$$

As we also have $f_\epsilon^{-\ell} \circ \Sigma_{2,\epsilon} = \Sigma_{2,\epsilon} \circ f_\epsilon^\ell$ we finally get $\Sigma_{2,\epsilon}(z_{\ell+1}) = z_{\ell+1}$.

In both cases let $z_{\ell+1} = z_s$. If $\Sigma_1(z_s) = z_s$, then

$$\tilde{\Sigma}_{2\ell+1,\epsilon}(f_\epsilon^\ell(z_s)) = f_\epsilon^{2\ell} \circ \Sigma_1 \circ f_\epsilon^\ell(z_s) = f_\epsilon^\ell \circ \Sigma_1(z_s) = f_\epsilon^\ell(z_s). \tag{5.41}$$

If $\Sigma_{2,\epsilon}(z_s) = z_s$, then

$$\begin{array}{rcl} \tilde{\Sigma}_{2\ell+2,\epsilon}(f_\epsilon^\ell(z_s)) & = & f_\epsilon^{2\ell} \circ (\Sigma_{2,\epsilon} \circ \Sigma_1) \circ \Sigma_1 \circ f_\epsilon^\ell(z_s) = f_\epsilon^{2\ell} \circ \Sigma_{2,\epsilon} \circ f_\epsilon^\ell(z_s) \\ & = & f_\epsilon^\ell \circ \Sigma_{2,\epsilon}(z_s) = f_\epsilon^\ell(z_s). \end{array} \tag{5.42}$$

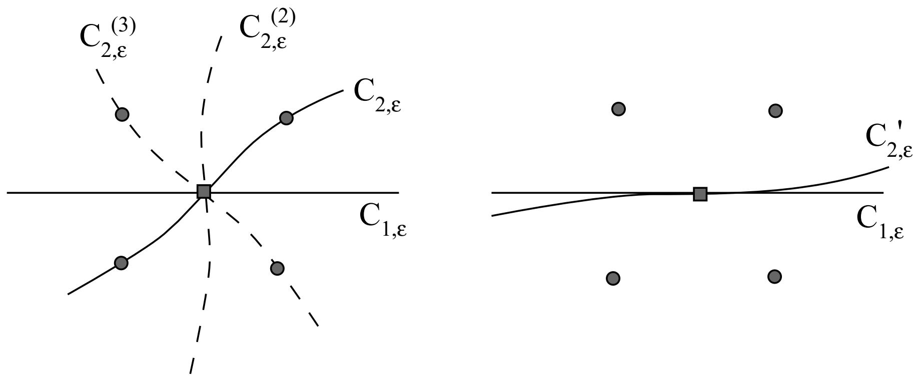


Figure 8: Same as Figure 7 with other values of ϵ

□

If the angle $(C_{1,\epsilon}, C_{2,\epsilon})$ could be sected in N parts, (i.e. f_ϵ would have an N-th root g_ϵ), additional periodic points are needed (see Figure 9). For instance, in the case $N = 2$ as in

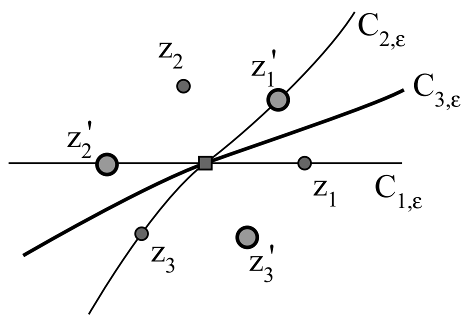


Figure 9: The need for new additional periodic points in case of a bisection of the angle

Figure 9, let $z'_1 = \Sigma_{3,\epsilon}(z_1)$. If $z_1 \in C_{1,\epsilon}$, then $z'_1 \in C_{2,\epsilon}$. As $g_\epsilon = \Sigma_{3,\epsilon} \circ \Sigma_1$ then $g_\epsilon(z_1) = z'_1$. Hence $f_\epsilon^{mn+1}(z'_1) = z'_1$, i.e. z'_1 is a point of an orbit of period $mn+1$. The other points of the orbit are $z'_j = f_\epsilon^{j-1}(z'_1)$. We get a contradiction as in Theorem 4.1 since there are only $1 + mn$ periodic points, while $N(1+mn)$ periodic points would be needed if the section was possible.

(ii) External bisection of the zero curvilinear angle. We need to find a function $g_0(z) = -z + o(z)$ such that $g_0^2(z) = f_0(z)$, where $f_0(z) = z + iz^2 + o(z^2)$. There is a formal obstruction to this as $g_0^2(z) = z + o(z^2)$ (the first higher order coefficient of g_0 is of odd order). External bisection means for instance finding $C_{3,0}$ making an angle $\pi/2$ with $C_{1,0}$ (and $C_{2,0}$).

The obstruction to an external section of the angle comes from the fact that the multiplicity of intersection of $C_{1,0}$ and $C_{2,0}$ is only 2 while a multiplicity of order 3 is necessary for an external section. Indeed suppose that g_0 exists and suppose that g_0 can be unfolded as g_ϵ, which means that we have unfoldings $C_{j,\epsilon}$ of the curves $C_{j,0}$, $j = 1, 2, 3$. Because $C_{3,0}$ and $C_{1,0}$ are orthogonal, then necessarily $C_{3,\epsilon}$ and $C_{1,\epsilon}$ have a unique intersection point, yielding a unique fixed point of g_ϵ. So all other fixed points of f_ϵ come from periodic points of g_ϵ of period 2, so there are an even number of them. Hence f_ϵ always has an odd number of fixed points.

On the other hand, suppose now that, instead of considering a generic f_0, we consider $f_0(z) = z \pm iz^3 + o(z^3)$ (i.e. f_0 of codimension 2) , then its modulus is given by 4 germs of functions $\psi_j^{0,\infty}$, $j = 0, 1$. The condition $f_0 = g_0^2$ means that the four germs are related by the following relations

$$\psi_1^{0,\infty}(w) = -\psi_0^{0,\infty}(-w), \tag{5.43}$$

i.e. the modulus is given by the pair $(\psi_0^0, \psi_0^\infty)$ [4]. This case has been discussed in Theorem 4.3, where it was shown that when the condition (5.43) is violated there are periodic points which are obstructions to the bissection problem. We do not discuss it any longer.

(iii) Bisection of a rational nonzero curvilinear angle. We consider a generic f_0 as in (5.12). If p is odd or if p is even and we consider an external bisection $h_0(z) = \exp(2\pi i\frac{p}{2q} + \pi i)z + o(z)$ then this case is completely similar to the previous one: The bisection is formally impossible as we should have $f_0^q(z) = h_0^{2q}(z) = z + O(z^{2q+1})$. Such a bisection would be possible for f_0 of the form $f_0(z) = \exp(2\pi i\frac{p}{q})z + C\frac{p}{q})z^{2q+1} + o(z^{2q+1})$ provided the modulus $(\psi_j^{0,\infty})$, $j = 0, \dots 2q-1$, would satisfy

$$\psi_{\sigma(j)}^{0,\infty}(w) = \exp\left(-\frac{\pi i}{q}\right)\psi_j^{0,\infty}\left(\exp\left(\frac{\pi i}{q}\right)w\right) \tag{5.44}$$

where $\sigma(j) \equiv j+p \pmod{2q}$ (resp. $\sigma(j) \equiv j+p+q \pmod{2q}$) for the internal (resp. external) bisection. The case of p even and an internal bisection $h_0(z) = \exp(2\pi i\frac{p}{2q})z + o(z)$ will be considered in (iv) below.

(iv) The case of the internal section in N parts of an angle $2\pi\frac{p}{q}$ when $N|p$. This case is completely similar to (i) and covered by Theorem 4.2. The two curves $C_{1,\epsilon}$ and $C_{2,\epsilon}$ always have a unique intersection point. Multiplying the angle by q the curves $C_{1,\epsilon}$ and $C'_{2,\epsilon}$ have one to three intersection points (the other solutions do not belong to $\mathrm{Fix}(\Sigma)$). The obstruction to the section in N equal parts can again be explained in terms of cascades or periodic points.

6 The modulus of orbital analytic classification of generic 1-parameter families of real vector fields unfolding a resonant saddle or saddle-node

Here again we only discuss very briefly the codimension 1 case. It has been shown in [14] for the resonant saddle case (resp. [13] for the saddle-node case) that a complete modulus of orbital analytic classification for a generic 1-parameter family of vector fields unfolding a resonant saddle (resp. saddle-node) is given by a complete modulus of analytic classification for the family of holonomies of one separatrix (resp. the strong separatrix). Let f_ϵ be this family. If we have a family of real analytic vector fields it is easy to verify that f_ϵ satisfies (5.13). Hence Theorem 5.6 applies in this case and characterizes the moduli of orbital analytic classification of generic 1-parameter families of real vector fields.

7 Conclusion

Theorem 4.1, Theorem 4.2 and Theorem 4.3 only explain the obstructions to the existence of an N-th root near some sequences of parameter values (ϵ_n). For these parameters values some of the fixed or periodic points are resonant. For the other values of ϵ (in particular the values of ϵ in the Poincaré domain) it occurs often that there exists an N-th root of f_ϵ near each of the fixed or periodic points. The obstruction to a section over a fixed neighborhood containing all the fixed points is then an incompatibility for the local N-th roots to glue in a global N-th root. The particular sequences of parameters we have chosen are those where the obstructions are carried by the fixed or periodic points themselves (the parametric resurgence phenomenon).

The case of fixed points with non resonant multiplier on the unit circle is particularly interesting. Indeed a germ of diffeomorphism $\tilde{f}$ with a fixed point whose multiplier is of the form $\exp(2\pi i\alpha)$ where α is Liouvillian, may not have an N-th root. Such diffeomorphisms occur in the unfolding f_ϵ of a germ of diffeomorphism f with a fixed point whose multiplier is a root of unity. Can we find conditions on the Ecalle-Voronin modulus of f guaranteeing that in all unfoldings f_ϵ the map f_ϵ would have no N-th root in a neighborhood of a fixed point whose multiplier is of the form $\exp(2\pi i\alpha)$ where α is Liouvillian? Of course "Liouvillian" would need to be defined for this special problem. The author conjectures that the answer is negative.

8 Acknowledgements

The author is grateful to Colin Christopher and Yulij Ilyashenko for helpful discussions.

References

[1] P. Ahern and X. Gong, A complete classification for pairs of real analytic curves in the complex plane with tangential intersection, preprint, University of Wisconsin, Madison, 2005.

[2] C. Christopher and C. Rousseau, Analytic moduli for unfoldings of real 2-dimensional saddles and weak foci, in preparation.

[3] J. Ecalle, Les fonctions résurgentes, *Publications mathématiques d'Orsay* (1985).

[4] Y. Ilyashenko, Nonlinear Stokes phenomena, in *Nonlinear Stokes phenomena*, Y. Ilyashenko editor, Advances in Soviet Mathematics, vol. 14, Amer. Math. Soc., Providence, RI, (1993), 1-55.

[5] E. Kasner, Conformal geometry, *Fifth Int. Cong. Math. Cambridge* **2** (1912), 81–87.

[6] E. Kasner, Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry, *Trans. Amer. Math. Soc.* **16** (1915), 333–349.

[7] J. Martinet and J.-P. Ramis, Classification analytique des équations différentielles non linéaires résonantes du premier ordre, *Ann. Sci. École Norm. Sup.*, **16** (1983), 571–621.

[8] P. Mardešić, R. Roussarie and C. Rousseau, Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms, *Moscow Mathematical Journal*, **4** (2004), 455–502.

[9] I. Nakai, The classification of curvilinear angles in the complex plane and the group of $\pm$ holomorphic diffeomorphisms, *Annales de la Faculté des Sciences de Toulouse*, **7** (1998), 313–334.

[10] G.A. Pfeiffer, On the conformal geometry of analytic arcs, *Amer. J. Math.* **37** (1915), 395–430.

[11] G.A. Pfeiffer, On the conformal mapping of curvilinear angles. The functional equation $\phi[f(x)] = a_1\phi(x)$, *Trans. Amer. Math.Soc.* **18** (1917), 185–198.

[12] G.A. Pfeiffer, The functional equation $f[f(x)] = g(x)$, *Annals of Mathematics*, **20** (1918), 13-22.

[13] C. Rousseau, Modulus of orbital analytic classification for a family unfolding a saddle-node, *Moscow Mathematical Journal* **5** (2005), 243–267.

[14] C. Rousseau and C. Christopher, Modulus of analytic classification for the generic unfolding of a codimension one resonant diffeomorphism or resonant saddle, preprint CRM (2004).

[15] J.-C. Yoccoz, Théorème de Siegel, nombres de Bruno et polynômes quadratiques, *Astérisque*, **231** (1995), 3–88.