Modulus of orbital analytic classification for a family unfolding a saddle-node

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Abstract

In this paper we consider generic families of 2-dimensional analytic vector fields unfolding a generic (codimension 1) saddle-node at the origin. We show that a complete modulus of orbital analytic classification for the family is given by an unfolding of the Martinet-Ramis modulus of the saddle-node. The Martinet-Ramis modulus is given by a pair of germs of diffeomorphisms, one of which is an affine map. We show that the unfolding of this diffeomorphism in the modulus of the family is again an affine map. The point of view taken is to compare the family with the "model family" $(x^2-\epsilon)\frac{\partial}{\partial x}+y(1+a(\epsilon)x)\frac{\partial}{\partial y}$. The nontriviality of the Martinet-Ramis modulus implies geometric "pathologies" for the perturbed vector fields, in the sense that the deformed family does not behave as the standard family.

1 Introduction

A vector field with a codimension 1 saddle-node at the origin is formally orbitally equivalent to the vector field

$$x^2\frac{\partial}{\partial x}+y(1+ax)\frac{\partial}{\partial y} \tag{1.1}$$

which we call the "model", but generically the change of coordinates and scaling of time to the form (1.1) diverges. **Why?** Because the geometry of the model is simpler than the geometry of the original vector field. The Martinet-Ramis modulus allows to describe the geometry of the foliation of the original vector field. Indeed the space of leafs is described on two open sectors whose union covers a neighborhood of the origin. In the model we have a trivial glueing, while the glueing is non trivial in generic saddle-nodes. The equivalence class of a saddle-node (under orbital equivalence) is completely characterized by the conjugacy class of the holonomy map of its strong separatrix, the later being given by the Ecalle-Voronin modulus of the holonomy map. In general the Ecalle-Voronin of a diffeomorphism having a generic parabolic fixed point at the origin is given by a pair of germs of diffeomorphisms on $\mathbb{CP}^1$, (ψ^0,ψ^∞), defined respectively at the origin and at infinity. In the particular case of the holonomy map of a saddle-node the map ψ^∞ is an affine map.

The following remark appears in [8]:

Un phénomène qui reste un peu surprenant à nos yeux est que les holonomies produites par les équations (2) ne sont pas arbitraires: on obtient seulement une "petite partie du module d'Ecalle". (Nous nous proposons de montrer dans un article ultérieur qu'il n'en est plus de même dans le cas des équations résonantes "non dégénérées": le module des classes d'équivalence analytiques d'équations différentielles s'identifie complètement au "module d'Ecalle").

Our study below will in particular provide a full explanation of this surprising phenonmenon. Indeed the best way to understand the geometric meaning of these non trivial glueings is by unfolding the saddle-node. This point of view has already been studied by Glutsyuk in [3] where he studies the unfolding in the Poincaré domain and shows that the Martinet-Ramis modulus is the limit of the transition maps between the linearizing changes of coordinates in the neighborhood of the two singular points. But this does not explain the particular form of one of the diffeomorphisms in the Martinet-Ramis invariant. We push his study further by considering also the unfolding in the Siegel direction when the point is deformed into a saddle and a node.

In this paper we treat the following questions:

(1) We show that a generic family of 2-dimensional vector fields unfolding a saddle-node can be brought by an analytic change of coordinate together with an analytic scaling of time, both depending analytically of the parameter ϵ, to the prenormal form:
$$\begin{aligned} \dot{x} &= x^2-\epsilon \\ \dot{y} &= (x^2-\epsilon)f_0(x)+y(1+a(\epsilon)x)+o(y). \end{aligned} \tag{1.2}$$

(2) We prove an analytic center manifold theorem for an analytic family (1.2) unfolding a saddle-node: we get a family of invariant manifolds which are analytic at the singular point $(-\sqrt{\hat{\epsilon}},0)$ and ramified at $(\sqrt{\hat{\epsilon}},0)$, for $\hat{\epsilon}$ is in an open sector of the universal covering of ϵ-space punctured at the origin, of opening greater than 2π.

(3) We give a geometric proof of the fact that the map ψ^∞ of the Martinet-Ramis modulus is an affine map. Indeed, by looking at the geometry of the leaves, we show that this map is necessarily a global diffeomorphism of $\mathbb{CP}^1$ fixing ∞. Hence it is an affine map as the affine maps are the only global diffeomorphims of $\mathbb{CP}^1$ fixing ∞.

(4) We show that the unfolded diffeomorphism ψ_ϵ^∞ remains global for the unfolding of the saddle-node, yielding that it remains an affine map.

(5) We exploit this to construct changes of coordinates (ramified at $\pm\sqrt{\epsilon}$) transforming the family (1.2) to the model family
$$\begin{aligned} \dot{x} &= x^2-\epsilon \\ \dot{y} &= y(1+a(\epsilon)x). \end{aligned} \tag{1.3}$$
This allows to show that the complete modulus of analytic classification for the holonomy map of the family (1.2) (see [10]), given by the family $(\psi_{\hat{\epsilon}}^0,\psi_{\hat{\epsilon}}^\infty)_{\hat{\epsilon}\in V}$ where V is an open sector of the universal covering of ϵ-space punctured at the origin, of opening greater than 2π, is a complete modulus of analytic classification for the family (1.2) under weak orbital equivalence (to be defined below).

(6) The non triviality (i.e. nonlinearity) of ψ^∞ implies that in the unfolding the analytic weak separatrix of the saddle is always ramified at the node. The latter implies that the node is non linearizable as soon as resonant. Similarly the non triviality (i.e. nonlinearity) of ψ^0 implies that the saddle is non integrable for some sequences of parameter values depending on the nonlinearities of ψ^0 (see also [10]). We call this phenomenon *parametric resurgence* as the pathologies of the system may only be seen on the types of the singular points at discrete values of the parameters.

(7) We finally treat an example: the Ricatti equation and its subcase, the linear equation. In this particular case we completely calculate ψ_ϵ^∞ and show that its coefficient is holomorphic if $a(\epsilon) \equiv n$, $n \in \mathbb{Z}$ and a quotient of two holomorphic functions with essential singularities at the origin otherwise. In the case of the linear equation we give the modulus space.

This paper is dedicated to Yulij Ilyashenko who first introduced the author to the Martinet-Ramis modulus.

2 Preparation of the family

We start with a system which has a generic saddle-node at the origin. Under an analytic change of coordinate and analytic scaling of time we can bring the system to the prenormal form

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= y(1+ax) + x^2 f_0(x) + x^2 y^2 R(x,y). \end{aligned} \tag{2.1}$$

We consider a generic family unfolding a saddle-node. If the family depends continuously on a parameter ϵ the following theorem was proved by Glutsyuk.

Theorem 2.1 *[3] There exists an analytic change of coordinates and analytic scaling of time, both depending continuously on the parameter ϵ, bringing a generic family unfolding a saddle-node to the prenormal form*

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= (x^2-\epsilon) g_0(x) + y(1+g(x,y,\epsilon)) \end{aligned} \tag{2.2}$$

where $g(x,y,\epsilon) = O(x) + O(y)$.

We improve his result. We start with a generic family unfolding a saddle-node. This is done in two folds. First we remark that, if we apply the Glutsyuk argument to a family depending analytically on ϵ, the change of coordinate to the form (2.2) is analytic in ϵ. We then "prepare" the family so that the parameter becomes an analytic invariant.

Theorem 2.2 *We consider a generic 1-parameter family unfolding a saddle-node. There exists an analytic change of coordinates and analytic scaling of time, both depending analytically on ϵ, bringing the family to the prepared normal form*

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= (x^2-\epsilon) g_0(x) + y(1+a(\epsilon)x) + O(y^2). \end{aligned} \tag{2.3}$$

In this form the parameter ϵ and $a(\epsilon)$ are analytic invariants.

Proof. Glutsyuk first shows that the family can be brought by an analytic change of coordinate and analytic scaling of time to the form

$$\begin{array}{rcl}\dot{x} & = & x^2-\epsilon+yR(x,y,\epsilon)\\ \dot{y} & = & (x^2-\epsilon)g_0(x)+y(1+q(x,y,\epsilon)),\end{array} \tag{2.4}$$

with $q(x,y,\epsilon)=O(x,y,\epsilon)$. For $\epsilon\neq 0$ we work with the parameter $\sqrt{\epsilon}$ defined on the Riemann surface of $\sqrt{\epsilon}$. The strong separatrices of the singular points $(\pm\sqrt{\epsilon},0)$ are analytic curves $x=F^{\pm}_{\sqrt{\epsilon}}(y)$ with $F^{\pm}_{\sqrt{\epsilon}}(0)=\pm\sqrt{\epsilon}$, depending analytically on $\sqrt{\epsilon}$. Glutsyuk shows that the graphs of $F^{\pm}_{\sqrt{\epsilon}}$ are defined on neighborhoods of zero whose size is independent of ϵ. Then one performs the change

$$x_1=\sqrt{\epsilon}\frac{2x-F^{+}_{\sqrt{\epsilon}}(y)-F^{-}_{\sqrt{\epsilon}}(y)}{F^{+}_{\sqrt{\epsilon}}(y)-F^{-}_{\sqrt{\epsilon}}(y)} \tag{2.5}$$

which straightens both separatrices simultaneously. As the change of coordinates is analytic in all $\sqrt{\epsilon}\neq 0$ and bounded in the neighborhood of $\epsilon=0$ it is analytic in $\sqrt{\epsilon}$. Moreover, as it is invariant under $\sqrt{\epsilon}\mapsto-\sqrt{\epsilon}$, it depends analytically on ϵ. The system in the variables (x_1,y) has the form

$$\begin{array}{rcl}\dot{x}_1 & = & (x_1^2-\epsilon)Q(x_1,y,\sqrt{\epsilon})\\ \dot{y} & = & (x_1^2-\epsilon)\overline{g}_0(x_1)+y(1+q_1(x_1,y,\sqrt{\epsilon}))\end{array} \tag{2.6}$$

and depends analytically on ϵ. We have $Q(x_1,y,\epsilon)=1+O(x_1,y,\epsilon)=Q_1(x_1,\epsilon)+O(y)$. We scale time by dividing by $Q(x_1,y,\epsilon)$. The new equation in y has the form $y'=(x_1^2-\epsilon)\hat{g}_0(x_1)+yq_2(x_1,y,\epsilon)$, with $q_2(x_1,y,\epsilon)=A(\epsilon)+O(x_1,y)$. Simultaneous scaling in (x_1,ϵ,t) (where t is the time) allows to suppose that $A(\epsilon)=1$.

Then the new equation in y has the form (we do not change the names of (x_1,ϵ)) $y'=(x_1^2-\epsilon)\tilde{g}_0(x_1)+y(1+q_3(x_1,\epsilon)+O(y))$. By Kostov's theorem [5] a change of coordinates $(x_1,\epsilon)\mapsto(\overline{x},\overline{\epsilon})$ allows to bring $\frac{x_1^2-\epsilon}{1+q_3(x_1,\epsilon)}\frac{\partial}{\partial x_1}$ to the form $\frac{\overline{x}^2-\overline{\epsilon}}{1+a(\overline{\epsilon})\overline{x}}\frac{\partial}{\partial\overline{x}}$. Applying this to the system divided by $1+q_3(x_1,\epsilon)$ and then multiplying the obtained system by $1+a(\overline{\epsilon})\overline{x}$ yields the result, by remarking that, in Kostov's proof, $(\overline{x}^2-\overline{\epsilon})$ is a nonzero multiple of $(x_1^2-\epsilon)$. Let $\mu_\pm=\pm\frac{2\sqrt{\epsilon}}{1\pm a(\epsilon)\sqrt{\epsilon}}$ be the quotient of the eigenvalues of the linearized vector field at $(\pm\sqrt{\epsilon},0)$. Then $\mu_\pm$ are analytic invariants of the system. The parameter ϵ and also $a(\epsilon)$ are analytic invariants since

$$\begin{array}{rcl}\frac{1}{\sqrt{\epsilon}} & = & \frac{1}{\mu_+}-\frac{1}{\mu_-}\\ a(\epsilon) & = & \frac{1}{\mu_-}+\frac{1}{\mu_+}.\end{array} \tag{2.7}$$

□

3 A center manifold theorem for analytic families of vector fields

To give a precise statement of the theorem we will recall briefly the notion of 1-summability of a power series $\sum_{n=0}^{\infty}a_nx^{n+1}$ (see for instance [1] or [8] for more details).

Definition 3.1 A power series

$$\sum_{n=0}^{\infty} a_n x^{n+1} \tag{3.1}$$

is 1-summable if

i) the series $\sum \frac{a_n \zeta^n}{n!}$ is convergent in a ball of radius r with analytic sum equal to $f(\zeta)$;

ii) the function $f(\zeta)$ extends analytically to $\mathbb{C} \setminus \Sigma$ where Σ is a finite union of half-lines

$$\Delta_j = \{\zeta | \arg \zeta = \theta_j, |\zeta| > \rho_i > 0\}; \tag{3.2}$$

iii) the function $f(\zeta)$ has at most exponential growth in $\mathbb{C} \setminus \Sigma$.

Consequences of Definition 3.1. Let $d = \{re^{i\theta_0} | r \in [0, \infty)\}$ be a half-line starting from the origin and contained in $\mathbb{C} \setminus \Sigma$. Then the integral

$$g_d(z) = \int_d e^{-\frac{\zeta}{z}} f(\zeta) d\zeta \tag{3.3}$$

converges on a sector of small radius centered on d, of opening of almost π: more precisely, for each $\delta > 0$ arbitrarily small there exists $r(\delta) > 0$ such that the function g_d is defined on $|z| < r(\delta)$, $\arg(z) \in (\theta_0 - \pi/2 + \delta, \theta_0 + \pi/2 - \delta)$. The function g_d is called a *sum* of the series (3.1). If we deform continuously the half-line d in $\mathbb{C} \setminus \Sigma$, the corresponding functions g_d are analytic continuations of each other. So, altogether, if n is the number of half-lines Δ_j in Σ, the formula (3.3) for all half-lines in $\mathbb{C} \setminus \Sigma$ yields n functions $g_1, \dots, g_n$, which are sums of the series (3.1).

On the intersection of the domains of definitions of g_j and g_k there exist $C, K > 0$ such that:

$$|g_j(z) - g_k(z)| < C \exp\left(-\frac{K}{|z|}\right). \tag{3.4}$$

Theorem 3.2 *We consider a germ of family of analytic vector fields of the form*

$$v_\epsilon = (x^2 - \epsilon)\frac{\partial}{\partial x} + \left[(x^2 - \epsilon) f_0(x, \epsilon) + y(1 + a(\epsilon)x) + \sum_{i \geq 2} f_i(x, \epsilon) y^i\right] \frac{\partial}{\partial y}. \tag{3.5}$$

Let $\delta > 0$ small. Then there exists a sector V of the universal covering of $\mathbb{C} \setminus \{0\}$:

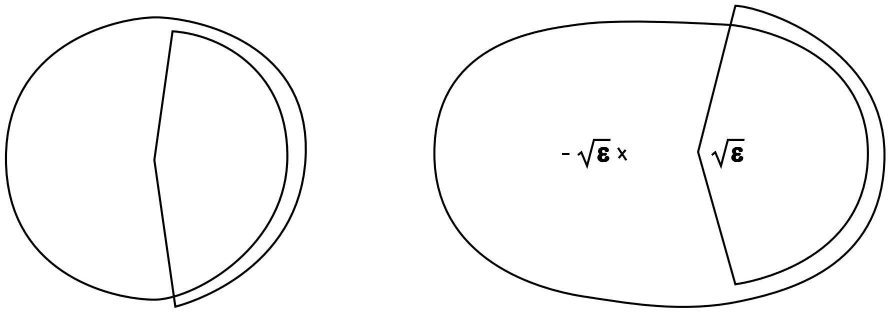
$$V = \{\hat{\epsilon}; |\hat{\epsilon}| < \rho, \arg \hat{\epsilon} \in (-2\pi + \delta, 2\pi - \delta)\}, \tag{3.6}$$

a neighborhood U of the origin in x-space and an invariant manifold defined as the graph of a function $y = g_{\hat{\epsilon}}(x)$ defined on a neighborhood U of the origin in x-space such that

i) For $\epsilon = 0$ the function $g_0(x)$ is the sum of a power series which is 1-summable in all directions except in the direction $\mathbb{R}^+$ and its sum is a ramified function over U (Figure 1a).

ii) The function $g_{\hat{\epsilon}}(x)$ depends analytically on $\hat{\epsilon} \neq 0$ and continuously on $\hat{\epsilon}$ near $\epsilon = 0$.

iii) The function $g_{\hat{\epsilon}}(x)$ is analytic on U minus a cut from $\sqrt{\epsilon}$.



(a) $\epsilon = 0$ (b) $\hat{\epsilon} \neq 0$

Figure 1: The domain of $g_{\hat{\epsilon}}$

iv) As a ramified function the function $g_{\hat{\epsilon}}$ is defined over a domain of the form $U_{\hat{\epsilon}}$ projecting on U as in Figure 1b).

Proof. We can scale y and suppose that the system (3.5) is defined over $|y| < 2$.

The function $g_\epsilon(x)$ of the theorem must be a solution of the nonlinear differential equation:

$$(x^2 - \epsilon)g'_\epsilon(x) = g_\epsilon(x)(1 + ax) + \sum_{i\geq 2} f_i(x,\epsilon)g^i_\epsilon(x) + (x^2 - \epsilon)f_0(x,\epsilon). \tag{3.7}$$

For $\epsilon = 0$ the solution of (3.7) is 1-summable in all directions except in the direction $\mathbb{R}^+$ [8]. This yields to the existence of a ramified solution over a domain U_0 in the universal covering of $\mathbb{C} \setminus \{0\}$ as in Figure 1a), the opening of the sector being $(-\pi/2+\delta, 5\pi/2-\delta)$. We take U_1 to be the projection of U_0. We suppose that U_1 contains a disk $\overline{U} = \{|x| \leq r\}$ with interior U. Let $0 < R < 1$. We can always suppose that r is sufficiently small so that $|g_0(x)| < |x| < R$ for $|x| = r$ (this comes from the fact that $g_0(x)$ has an asymptotic expansion of the form $O(x^2)$ near $x = 0$.

For $\epsilon \in V$ the equation (3.7) has an analytic solution defined in the neighborhood of $-\sqrt{\epsilon}$ and vanishing at $x = -\sqrt{\epsilon}$ (because the quotient of eigenvalues is not a positive real number). For $|\epsilon|$ sufficiently small we now need to extend this solution to the fixed neighborhood $|x| \leq r$, independently of ϵ, with a cut from $x = \sqrt{\epsilon}$. The ideas used here are borrowed from Glutsyuk [3]. Indeed for (x,ϵ) sufficiently small the inequality $|\dot{y}| > |\dot{x}|$ is satisfied for (x,y) in the cones: $K_1(\epsilon) = \{(x,y); |y| > |x-\sqrt{\epsilon}|\}$ and $K_2(\epsilon) = \{(x,y); |y| > |x+\sqrt{\epsilon}|\}$. Also leaves of the foliation of (3.5) contain trajectories with real time of all systems of the form

$$v_\epsilon(\theta) = e^{i\theta} \times v_\epsilon. \tag{3.8}$$

We need to find points $(x_1, \overline{g}_\epsilon(x_1))$, with $|x_1| = r$, which "should" belong to the center manifold and are located under the cones K_1 and K_2. The extension of their trajectories under the different $v_\epsilon(\theta)$ will yield the full center manifold. The details are as follows.

We let $x_0 = -r$. Let ϕ^t_0 be the flow of v_0 (we allow complex time). Then for all $(x_1, g_0(x_1))$ with $|x_1| = r$ and $\arg(x_1) \in (-\pi/2+\delta, 5\pi/2-\delta)$ there exists $t(x_1) \in \mathbb{C}$ such that $(x_1, g_0(x_1)) = \phi_0^{t(x_1)}(x_0, g_0(x_0))$ ($t(x_1)$ is multivalued on $\Re x_1 > 0$.)

Let $\eta > 0$ small. The trajectories with real time starting at (x_0, y) with $|y - g_0(x_0)| = \eta$, i.e on a circle Γ, cross the cylinder C given by $|y| = 1$ along a non-contractible loop γ: this yields a continuous map P_0 from the circle Γ to the cylinder C.

We will limit ourselves to values of $|\epsilon| < \rho_1$ with ρ_1 sufficiently small so that $\pm\sqrt{\epsilon}$ remain inside $|x| < r$. For small ϵ the map P_0 is deformed to a continuous map P_ϵ from the circle Γ to the cylinder C. Hence there is a topological obstruction to the continuous extension of P_ϵ to the disk $D = \{x = x_0, |y - g_0(x)| \leq \eta\}$ given by the interior of Γ, yielding that the orbit of at least one point (x_0, y_ϵ) of D does not meet the cylinder.

Then the forward trajectory of (x_0, y_ϵ) "remains under" the cones K_1 and K_2, i.e. lies in the region $|y| < \min(|x - \sqrt{\epsilon}|, |x + \sqrt{\epsilon}|)$. We let $\overline{g}_\epsilon(x_0) = y_\epsilon$.

For all x_1 with $|x_1| = r$ there exists $t_\epsilon(x_1)$ ($t(x_1)$ is multivalued on $\Re x_1 > 0$) such that $\phi_\epsilon^{t_\epsilon(x_1)}(x_0, y_\epsilon) = (x_1, y_{1,\epsilon})$. We let $\overline{g}_\epsilon(x_1) = y_{1,\epsilon}$. When ϵ is small the map $\overline{g}_\epsilon$ is close to g_0 on $|x| = r$.

For all $\hat{\epsilon} \in V$ such that the x-eigenvalue of $v_{\hat{\epsilon}}$ at $(-\sqrt{\hat{\epsilon}}, 0)$ has a negative real part (for instance $\arg(\hat{\epsilon}) \in (-\pi + \delta, \pi - \delta)$) at least one of the trajectories of a $v_\epsilon(\theta)$ starting at one point $(x_1, y_{1,\epsilon})$ tends to $(-\sqrt{\hat{\epsilon}}, 0)$, yielding that it is part of the invariant manifold of that point. Hence all trajectories starting at points $(x_1, y_{1,\epsilon})$ belong to the invariant manifold of $(-\sqrt{\hat{\epsilon}}, 0)$, i.e. give an extension of $g_0(x)$.

The property follows for the other values of $\hat{\epsilon}$ by analytic extension. □

Theorem 3.3 *We consider a family* (3.5) *as in Theorem* 3.2 *which has no analytic center manifold for* $\epsilon = 0$. *Then the function* $g_{\hat{\epsilon}}(x)$ *is ramified in a nontrivial way at* $\sqrt{\epsilon}$ *for all* $\hat{\epsilon} \in V$. *In particular the node of* (3.5) *located at* $(\sqrt{\epsilon}, 0)$ *is non linearizable (i.e. has a nonzero resonant monomial) as soon as it is resonant.*

Proof. From our proof it follows that if the map g_0 of Theorem 3.2 is ramified over $|x| \leq r$, then so is the map $g_{\hat{\epsilon}}$ for small $\hat{\epsilon}$. The only point where the ramification can occur is $\sqrt{\hat{\epsilon}}$ as the map is analytic in x elsewhere. The second part follows by noting that all trajectories through a linearizable resonant node are analytic and therefore cannot be ramified. Hence the only possible way to have a non analytic trajectory passing through a resonant node is to have a nonlinearizable node (see also [10]). □

4 The unfolding of the Martinet-Ramis modulus

4.1 The Martinet-Ramis modulus of orbital analytic classification for a saddle-node

The description of the modulus of orbital analytic classification of a saddle-node, called the *Martinet-Ramis modulus* makes use of first integrals defined in sectorial neighborhoods of the saddle-node ([8] and [4]). Moreover it is shown in [8] that the orbital analytic class of a saddle-node is characterized by the analytic class of the holonomy of its strong separatrix (see also [4]). Indeed, restricted to an adequate domain, the first integral is a tool to describe the space of orbits of the vector field (leaves of the foliation), the later coinciding (up to isolated elements) with the space of orbits of the holonomy map.

The holonomy map is formally equivalent to the $2\pi i$-time map of the vector field $\frac{x^2}{1+ax}\frac{\partial}{\partial x}$

i.e. the time 1-map of

$$2\pi i \frac{x^2}{1+ax}\,\frac{\partial}{\partial x}, \tag{4.1}$$

(see Figure 2). Taking $\overline{x} = 2\pi i x$ we can also see it as the time one map of

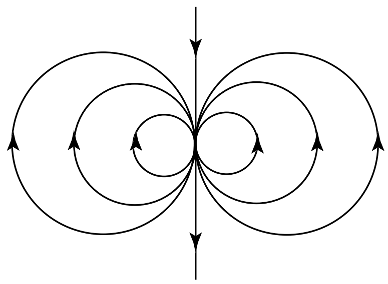


Figure 2: The vector field (4.1)

$$\frac{\overline{x}^2}{1+\overline{a}\,\overline{x}}\,\frac{\partial}{\partial \overline{x}}, \tag{4.2}$$

with $\overline{a} = \frac{a}{2\pi i}$.

For the holonomy map the space of orbits is described by two fundamental domains for sectorial neighborhoods $U^{\pm}$ of the origin where

$$\begin{aligned} U^+ &= \{x | \arg x \in (-3\pi/2 + \eta, \pi/2 - \eta), |x| < r\} \\ U^- &= \{x | \arg x \in (-\pi/2 + \eta, 3\pi/2 - \eta), |x| < r\}, \end{aligned} \tag{4.3}$$

with $\eta \in (0, \pi/2)$ (the smaller η, the smaller r). Once quotiented by the holonomy map, these fundamental domains have the conformal structure of punctured spheres. As orbits may have representatives in the two fundamental domains the spheres are glued in the neighborhoods of zero and infinity by the Ecalle-Voronin modulus (ψ^0, ψ^∞), where ψ^0 and ψ^∞ are germs of analytic diffeomorphisms in the neighborhood of 0 and ∞ (Figure 3). When unfolding a

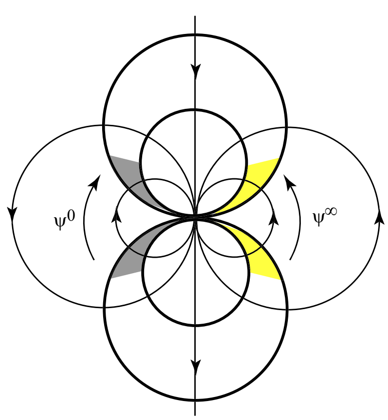


Figure 3: The fundamental domains and transition maps

saddle-node in the Siegel direction we will get essentially a saddle and a node ("essentially" because if $a \notin \mathbb{Z}$ there may be a small shift between the values of ϵ for which one point is a saddle and those for which the other singular point is a node). We choose to call ∞ (resp. 0) the point of the sphere which will be attached to the node (resp. saddle). The coordinates on the spheres are uniquely determined up to linear transformations on each sphere. Under rescaling of y we can suppose that the holonomy map is defined for a section $\{y = 1\}$: we call it f_0. All leaves of the foliation intersect this section except possibly one (the center manifold). Each leaf intersects at least one fundamental domain. Hence, it is natural to take the spherical coordinates as first integrals. Then the Ecalle-Voronin modulus represents exactly the transitions between the two first integrals $H_0^{\pm}$ defined on $U^{\pm} \times W$ where W is a neighborhood of the origin in y-space. These two first integrals are the "canonical" first integrals

$$H_0^{\pm}(x, Y) = Y_{\pm} x^{-a} e^{\frac{1}{x}} \tag{4.4}$$

for the model

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{Y} &= Y(1 + ax), \end{aligned} \tag{4.5}$$

where $Y_{\pm} = Y_{\pm}(x, y)$ are the normalizing coordinates on the two domains $U^{\pm} \times W$. (See for instance [4]).

Remark 4.1 Note that this first integral is in general multivalued, even for the model, as soon as $a \notin \mathbb{Z}$. However it can be defined in a single valued way in each of the domains $U^{\pm} \times W$.

In general all pairs of germs of analytic diffeomorphisms (ψ^0, ψ^∞) at the origin and at ∞ can be realized as the Ecalle-Voronin modulus of a germ of diffeomorphism with a generic parabolic fixed point or as the Martinet-Ramis modulus of a saddle with eigenvalues $1 : -1$. However in the particular case of the Ecalle-Voronin modulus of the holonomy of a saddle-node then the map ψ^∞ is always an affine map (a translation if we choose properly the coordinates on the spheres). In [10] we had shown that it was the only diffeomorphism which could be allowed in order to respect the properties of the node, namely to be linearizable as soon as resonant. Here we will give a more geometric explanation, using that the affine maps are the only global diffeomorphims of the sphere which have a fixed point at ∞.

Proposition 4.2 *The germ of diffeomorphism ψ^∞ of the Martinet-Ramis modulus of a saddle-node is an affine map (a translation if we scale ${\psi^\infty}'(\infty) = 1$).*

Proof. The idea of the proof is to show that ψ^∞ is a global diffeomorphism of $\mathbb{CP}^1$ fixing ∞. ψ^∞ exists as a local diffeomorphism in the neighborhood of ∞, with ∞ as a fixed point. We must show that it extends analytically to the whole of $\mathbb{CP}^1$ and that its extension is 1-1.

Using scaling in y, we can always suppose that the vector field is defined in a neighborhood $U \times W$ with $U = \{x; |x| < r\}$ and $W = \{y; |y| < 2\}$ and that the normalizing coordinates defined above exist on the whole of $U^{\pm} \times W$ whose union is $U \times W$.

As mentioned above and explained in [4], on $y = 1$ it is shown the first integrals $H_0^{\pm}$ (given in (4.4)) take all values in $\mathbb{C}^*$ on each fundamental domain of the holonomy map and hence provide (up to a linear transformation) the natural parametrizations of the fundamental domains by $\mathbb{CP}\setminus^1\{0,\infty\}$. The first integral $H_0^\pm$ is defined on the whole of $U^\pm\times\{y=1\}$. We can also extend analytically the spherical coordinate by asking that it be constant on orbits of the holonomy map. Hence this extension must be equal to $H_0^\pm$ on the whole of $U^\pm\times\{y=1\}$. This means that points of $U^\pm\times\{y=1\}$ on which $H_0^\pm$ takes the same value belong to the same orbit of the holonomy map.

So on each region $U^\pm\times\{y=1\}$ we have a bijection between the orbits of the holonomy map on $U^\pm$ and the values of $H_0^\pm$. We can see this as a bijection between the orbits of the holonomy map and the connected components of the level sets of $H_0^\pm$(leaves of the foliation) on $U^\pm\times W$ which intersect $\{y=1\}$,($H_0^\pm$ is univalued on this domain).

The two components of $U^+\cap U^-=U^0\cup U^\infty$ yield the maps ψ^0 and ψ^∞ near the end points of the fundamental domains. U^0 (resp U^∞) is located in $\Re x<0$ (resp. $\Re x>0$). We use the bijection above to construct the analytic extension of ψ^∞ outside its natural domain of definition on $U^\infty\times\{y=1\}$: indeed ψ^∞ is an identification of orbits of the holonomy map. This can be done by identifying orbits with connected components of the leaves of points in $U_1\times W_1$, where $W_1=\{y,|y|<1\}$ and $U_1=\{x;|x|<r_1<r,|\arg x|<\eta_0\}\subset U^\infty$ with $\eta_0\in(0,\pi/2)$.

To do the construction we need to understand how we relate a point (x,y) in a neighborhood of the origin, and hence lying on a leaf, to a point in a fundamental domain in $y=1$. Leaves are 1-dimensional surfaces in $\mathbb{C}^2$, so have real codimension 2, while trajectories with real time have real codimension 3. A leaf contains non only the real trajectories but also all real trajectories of any multiple $e^{i\theta}v$ of the initial vector field v. We will always limit ourselves to $\theta\in(-\pi/2,\pi/2)$ so as to keep a positive real part for the y-eigenvalue of $e^{i\theta}v$.

Starting from $(x_0,y_0)\in U_1\times W_1$ we associate to it a point $(x_2,1)$ of its leaf lying in $\{y=1\}$. If $x_2\in U^\pm$ this point is naturally associated to the fundamental domain for $U^\pm$. Indeed if $r_1>0$ is sufficiently small there exist $\theta_1\in(0,\pi/2)$ and $\theta_2\in(-\pi/2,0)$ such that γ_1 (resp. γ_2), the projection of the trajectory of (x_0,y_0) for $v_0(\theta_1)=e^{i\theta_1}v_0$, (resp. $v_0(\theta_2)=e^{i\theta_2}v_0$) onto the x-axis in included in a trajectory of $e^{i\theta_1}x^2\frac{\partial}{\partial x}$ (resp. $e^{i\theta_2}x^2\frac{\partial}{\partial x}$) tending to the origin in the region $\Re x<0$ and contained in $U^+\times W$ (resp. $U^-\times W$).

Except possibly if we are on the center manifold the trajectory through (x_0,y_0) will enter the cone $K=\{(x,y);|y|>|x|\}$ and cut the cylinder $|y|=1$ at a value (x_1,y_1) with $y_1=e^{i\theta_3}$, $\theta_3\in(0,2\pi]$. Following the curve $(x(\theta),y_1e^{i\theta})$ contained in the leaf for $\theta\in[\theta_3,2\pi]$ (as in the construction of the holonomy map) allows to find a point $(x_2,1)$ where the leaf through (x_0,y_0) cuts the section $y=1$. For a given x_0, the smaller y_0, the more negative $\Re x_2$.

The curves γ_1 and γ_2 are not homotopic in $U_1\setminus\{0\}$. Moreover γ_1 (resp. γ_2) allows to associate to (x_0,y_0) a point $(x_2^+,1)$ with $x_2^+\in U^+$ (resp. $(x_2^-,1)$ with $x_2^-\in U^-$), thus a representative of its orbit in the fundamental domain and all points of the fundamental domains are obtained with this method. Hence the procedure gives a surjection between $U_1\times W_1$ and each of the fundamental domains of Figure 4. The map ψ^∞ identifying orbits in each fundamental domain is thus a global diffeomorphism with a fixed point at ∞: it is then affine, and a translation for a good choice of coordinates on the spheres: $\psi^\infty(w)=w+C$. We now recover the geometric interpretation of C: if $C=0$ the center manifold is not ramified hence analytic, while if $C\neq 0$, because of its ramification, the center manifold will intersect $|y|=1$. □

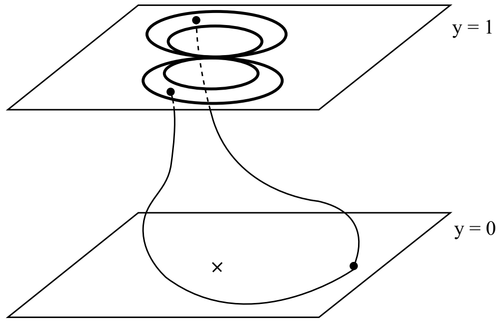


Figure 4: Construction of the first integral for $\epsilon = 0$

4.2 Unfolding of the Ecalle-Voronin modulus of the holonomy map of a saddle-node

We consider here a generic 1-parameter analytic family of diffeomorphisms unfolding a germ of diffeomorphism with a parabolic fixed point at the origin

$$f_\epsilon(x) = x + (x^2 - \epsilon)(1 + h(x, \epsilon)) \tag{4.6}$$

with $h(x, \epsilon) = O(x, \epsilon)$.

Definition 4.3 The family (4.6) is called in prepared form if the multipliers $\lambda_\pm = f'(\pm\sqrt{\epsilon})$ of the fixed points $\pm\epsilon$ are given by $\lambda_\pm = \exp(\mu_\pm)$, where $\mu_\pm$ are the eigenvalues of the linearized vector field

$$\frac{x^2 - \epsilon}{1 + a(\epsilon)x} \frac{\partial}{\partial x} \tag{4.7}$$

at the singular points $\pm\sqrt{\epsilon}$.

It is always possible to find an analytic change of coordinate and parameter $(x, \epsilon) \mapsto (\overline{x}, \overline{\epsilon})$ so as to transform a family (4.6) in prepared form (see details in [10]), in which case the new parameter is an analytic invariant.

The following theorem was shown in [10].

Theorem 4.4 *[10] A complete modulus of analytic classification of a germ of generic 1-parameter analytic family of diffeomorphisms* (4.6) *in prepared form is given by the family of unfoldings* $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon \in \tilde V}$ *of the Ecalle-Voronin modulus, together with* $a(0)$*, where:*

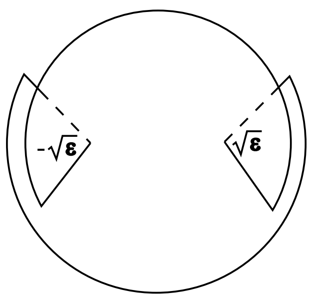
- $\psi^0_{\hat\epsilon}$, $\psi^\infty_{\hat\epsilon}$ *are germs of analytic diffeomorphisms;*
- *they depend analytically of* $\hat\epsilon$ *for* $\hat\epsilon \neq 0$ *and continuously on* $\hat\epsilon$ *near* $\epsilon = 0$*;*
- $\hat\epsilon$ *is in a sector* $\tilde V$ *of the universal covering of* ϵ*-space punctured at the origin;*
- *the sector* $\tilde V$ *has the form* $\arg\hat\epsilon \in (-\pi + \delta, 3\pi - \delta)$, $|\hat\epsilon| < \rho(\delta)$, *where* $\delta > 0$ *can be chosen arbitrarily small and* $\rho(\delta) > 0$ *depends on* δ *(and of the family).*

We now show that for the particular case of the holonomy map of the unfolding of a saddle-node, not only ψ^∞ is an affine map for $\epsilon = 0$, but its unfolding $\psi^\infty_{\hat\epsilon}$ remains an affine map for $\hat\epsilon \neq 0$. As before the geometric idea is that the map $\psi^\infty_{\hat\epsilon}$ must be global.

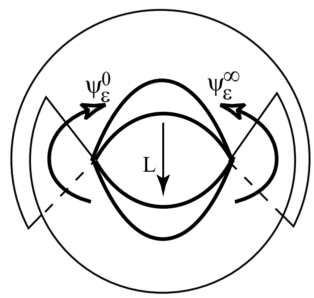
Remark 4.5 We need to consider the variable $\overline{x}$ of (4.2) and the corresponding formal invariant $\overline{a}$ if we want to apply Theorem 4.4 verbatim.

Theorem 4.6 *We consider a family* (2.3) *defined in a neighborhood of the origin of the form $U \times W$ where $W = \{y; |y| < 2\}$ and the holonomy of the unfolded system (which is a generic unfolding of the holonomy map of the strong separatrix) whose modulus of analytic classification is given by the family $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon \in V}$ unfolding the Ecalle-Voronin modulus of the holonomy map for $\epsilon = 0$. Then the map $\psi^\infty_{\hat\epsilon}$ is an affine map (a translation $w \mapsto w + C(\hat\epsilon)$ for adequate coordinates on the spheres).*

Proof. As in Proposition 4.1 we will show that the map $\psi^\infty_{\hat\epsilon}$ is a global diffeomorphism fixing ∞. The ideas of the proof are similar.



(a) The domain in x-space



(b) The Lavaurs map and transition maps

Figure 5: The domain in x-ccordinate

The Theorem 4.4 is proved by embedding for $\epsilon \neq 0$ the holonomy map into the time-1 map of the vector field (4.1) over two domains of the form as in Figure 5a. Fundamental domains are obtained as in Figure 5b. The dynamics here is completely different from that of the case $\epsilon = 0$. Indeed there exists a global analytic diffeomorphism L between the two fundamental domains, called the Lavaurs map, and identifying points with identical orbits. As L fixes 0 and ∞, it is a linear map. Instead of showing that $\psi^\infty_{\hat\epsilon}$ is a global analytic diffeomorphism it suffices to show the same property for the "renormalized return map" $L \circ \psi^\infty_{\hat\epsilon}$, which in practice compares orbits of points under the holonomy map when one turns around $\sqrt{\epsilon}$.

As $\psi^\infty_{\hat\epsilon}$ depends analytically on $\hat\epsilon$, it suffices to prove it is an affine map for $\hat\epsilon$ in a sector of small opening, for instance $\arg\hat\epsilon \in (-\delta, \delta)$ with $\delta > 0$ small. In this case the x-eigenvalue of $-\sqrt{\epsilon}$ (resp. $\sqrt{\epsilon}$) has a negative (resp. positive) real part.

We start with one fundamental domain in $\{y = 1\}$ which is parametrized by $\mathbb{CP}^1$, with $\sqrt{\epsilon}$ (resp. $-\sqrt{\epsilon}$) associated to ∞ (resp. 0), and we decide to consider the spherical coordinate on it as a first integral H. Iterating the holonomy map allows to extend H to $\{y = 1\}$ in a ramified way around $\pm\sqrt{\epsilon}$. If we turn around $\sqrt{\epsilon}$ in the positive (resp. negative) direction we obtain two branches $H^\pm$. We need to show that $H^+ = AH^- + B$ for $A \in \mathbb{C}^*, B \in \mathbb{C}$. To do that, as before, we extend H along the leaves of the foliation, so that it becomes a first integral.

As before we associate to points (x_0, y_0) with $|x_0| < r_1$, $|y_0| < 1$ and not on the stable manifold of $(-\sqrt{\epsilon}, 0)$, points of the section $y = 1$ which belong to the same leaf. This is done by associating to (x_0, y_0) a point of its leaf located on the cylinder $|y| = 1$ and pushing the construction of the holonomy map. As in Proposition 4.1 we consider points x_0 such that $|\arg(x_0 - \sqrt{\epsilon})| < \pi/4\}$. There exists $\theta_1 \in (0, \pi/4)$ and $\theta_2 \in (-\pi/4, 0)$ such that the trajectory of the 1-dimensional system $dx/dt = e^{i\theta_j}(x^2 - \epsilon)$ through x_0 with $|x_0| < r_1 < r$ makes no full turn around $\sqrt{\epsilon}$ and tends to $-\sqrt{\epsilon}$ by winding around it in the negative (resp. positive) direction for $j = 1$ (resp. $j = 2$), see Figures 6 and 7. Moreover the y-eigenvalues

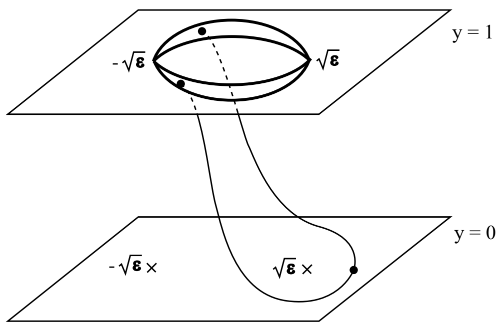
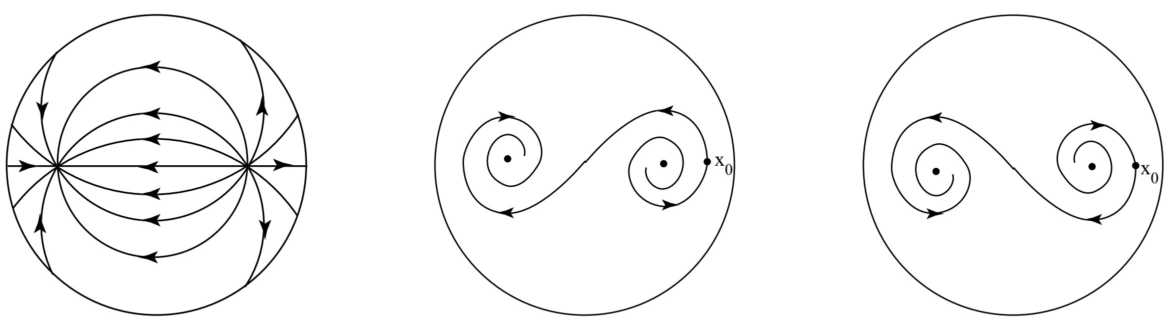


Figure 6: Construction of the first integral for $\epsilon \neq 0$



(a) v_ϵ for $\epsilon > 0$ (b) $v_\epsilon(\theta_1)$, $\theta_1 > 0$ (c) $v_\epsilon(\theta_2)$, $\theta_2 < 0$

Figure 7: The projection of trajectories of $v_\epsilon(\theta)$ on the x-axis

of the linearized vector field at $(\pm\sqrt{\epsilon}, 0)$ both have positive real parts. Hence trajectories of the corresponding vector fields $v_\epsilon(\theta_j)$ through (x_0, y_0) will intersect $|y| = 1$ except possibly if (x_0, y_0) is on the graph of the unfolding of the center manifold described in Theorem 3.2. We then associate to this point a point of the section $\{y = 1\}$ by applying the construction of the holonomy map. We finally iterate the holonomy map forwards or backwards (see explanation below) to end up with a point of the fundamental domain. The smaller $|y_0|$, the closer to $(-\sqrt{\epsilon}, 1)$ the associated point. Moreover all points of the fundamental domain for the holonomy map lie on trajectories of points (x_0, y_0) for the vector field $v_\epsilon(\theta_1)$ for some $\theta_1 > 0$ and on trajectories of points (x_0, y_0) for the vector field $v_\epsilon(\theta_2)$ for some $\theta_2 < 0$, for which the whole construction described above takes place over U minus cuts along two rays $\{x; |x| > |\sqrt{\epsilon}|, \arg x = \arg \pm\sqrt{\epsilon}\}$ (this last constraint determines if we iterate the holonomy forwards or backwards in the construction described above). Hence the transition map $\psi_{\hat{\epsilon}}^{\infty}$ is global. $\square$

4.3 Comparing the family (2.3) to the family model

The family model for the unfolding of a saddle node is given by

$$\begin{array}{rcl} \dot{x} & = & x^2 - \epsilon \\ \dot{Y} & = & Y(1 + a(\epsilon)x). \end{array} \tag{4.8}$$

For simplicity we will write a instead of $a(\epsilon)$. Its first integral is

$$\tilde{H}_\epsilon(x, Y) = Y(x^2 - \epsilon)^{-\frac{a}{2}} \left(\frac{x + \sqrt{\epsilon}}{x - \sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}} = Y F_\epsilon(x). \tag{4.9}$$

The integral is in general ramified at $x = \pm\sqrt{\epsilon}$.

The deformation in the Glutsyuk point of view. The point of view of Glutsyuk described in [3] and valid for a cone in ϵ-space avoiding the Siegel direction is that deformations H_ϵ^∞ (resp. H_ϵ^0) of the first integrals exist in neighborhoods of the singular points $(\sqrt{\epsilon}, 0)$ (resp. $(-\sqrt{\epsilon}, 0)$) with the following properties:

- They are the "canonical integrals" of the form $(x_1 - \sqrt{\epsilon})^{\frac{1}{\nu^+}} y_1$ and $(x_1 + \sqrt{\epsilon})^{\frac{1}{\nu^-}} y_1$ where (x_1, y_1) are linearizing coordinates at $(\pm\sqrt{\epsilon}, 0)$, so that $x_1 = \pm\sqrt{\epsilon}$ and $y_1 = 0$ are the analytic separatrices of the singular points and $\nu^\pm$ are the quotients of eigenvalues at $\pm\sqrt{\epsilon}$.
- H_ϵ^∞ (resp. H_ϵ^0) tends to H_0^+ (resp. H_0^-) as $\epsilon \to 0$.
- The transition between H_ϵ^∞ and H_ϵ^0 tends to the Martinet-Ramis modulus (the domain becomes disconnected at the limit).

We will not give more details on the Glutsyuk point of view and will concentrate on the "Lavaurs point of view", which consists in comparing the family (2.3) to the model family (4.8) "between the singular points" and to read the incompatibility of a full comparison when turning around the singular points.

The deformation in the Lavaurs point of view.

Theorem 4.7 *We consider a prepared family* (2.3)*. Then for $\hat{\epsilon} \in V$, where V is given in* (3.6)*, there exists a change of coordinate $Y = Y(x, y, \epsilon)$, holomorphic in y and in $x \neq \pm\sqrt{\epsilon}$, and ramified in x at $\pm\sqrt{\epsilon}$ (see Figure 5a)) bringing the system* (2.3) *to the model family* (4.8)*. The change of coordinate is holomorphic in $\hat{\epsilon}$ for $\hat{\epsilon} \neq 0$ and continuous in $\hat{\epsilon}$ near $\hat{\epsilon} = 0$. Moreover, near $\pm(\sqrt{\epsilon}, 0)$, $Y(x, y, \epsilon)$ has an asymptotic expansion $Y(x, y, \epsilon) = y + \sum_{i=0}^{\infty} b_{i,\epsilon}(x) y^i$ where $\lim_{x \to \pm\sqrt{\epsilon}} b_{0,\epsilon}(x) = 0$.*

Proof. The proof starts as that of Theorem 4.6. From a spherical coordinate on a fundamental domain of the holonomy map on $\{y=1\}$ we define a first integral for the system in the neighborhood of the origin. Each first integral can be continued (in a ramified way) on all $U\times\{y=1\}$ by asking that it be constant on orbits of the holonomy map. This first integral is ramified for $x=\pm\sqrt{\epsilon}$. The transition from one branch to the other can be seen through the maps $\psi^0_{\hat\epsilon}\circ L$ when turning around $-\sqrt{\epsilon}$ (resp. $\psi^\infty_{\hat\epsilon}\circ L$ when turning around $\sqrt{\epsilon}$), where L is the Lavaurs map (see Figure 5b). Comparing to the first integral (4.9) of the model this suggests the change of coordinate

$$Y(x,y,\epsilon)=\frac{H(x,y,\epsilon)}{F_\epsilon(x)}. \tag{4.10}$$

To finish the proof we first need to extend the construction of the first integral to a neighborhood $U\times W_1$. We also have to check that we can extend the proof of the construction of the first integral done only for a small sector in $\hat\epsilon$-space in Theorem 4.6 to the full sector V. We then have to check that $H(x,y,\epsilon)$ has the required asymptotic expansion allowing the conclusions of the theorem for the asymptotic expansion of $Y(x,y,\epsilon)$.

We first extend the first integral H to $\{|y|=1\}$: this is done as in the proof of Proposition 4.2. To extend it to $\{|y|<1\}$ we reproduce verbatim the construction of the first integral in the proof of Theorem 4.6 (see also Figure 4 and 6). Indeed as $\hat\epsilon\in V$ it is always possible to choose $\theta_1\in(0,\pi/2)$ or $\theta_2\in(-\pi/2,0)$ such that $e^{i\theta_1}v_\epsilon$ or $e^{i\theta_2}v_\epsilon$ have singular points with eigenvalues in y with positive real part and such that the eigenvalue in x has negative (resp. positive) real part at $x=-\sqrt{\epsilon}$ (resp. $x=\sqrt{\epsilon}$). In particular we get that any of the fundamental domains is completely covered by points belonging to leaves starting in the neighborhood of $(\sqrt{\epsilon},0)$.

We let $H(x,y,\epsilon)=0$ for $y=g_{\hat\epsilon}(x)$ and for x in a neighborhood of $-\sqrt{\epsilon}$. The H obtained in this way is obviously analytic in y and ramified in x at $x=\pm\sqrt{\epsilon}$.

We must now study the asymptotic properties of H near $(x,y)=(\pm\sqrt{\epsilon},0)$. For $\epsilon=0$ it corresponds to a first integral (4.9) constructed by means of a change of coordinate $y\mapsto Y(y,x,\epsilon)$ bringing the system (2.3) to the family model (4.8).

Near $\sqrt{\epsilon}$ we compare our integral with a standard integral of the form

$$H_1(x,y_1,\epsilon)=\frac{y_1-k_{\hat\epsilon}(x)}{(x-\sqrt{\epsilon})^{\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}} \tag{4.11}$$

where $y\mapsto y_1$ is a change of coordinate bringing the node to normal form and $y_1-k_{\hat\epsilon}(x)=0$ is the trajectory $y-g_{\hat\epsilon}(x)=0$ through $\sqrt{\epsilon}$ written in the coordinate y_1 and hence possibly ramified at $x=\sqrt{\epsilon}$. We get a bijection between the leaves of the foliation in a sector neighborhood of the node and the values of H on one side, the values of H_1 on the other side. (We could be surprised that H can take all values in $\mathbb{CP}^1$ when $\Re\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}<0$. This comes from the form of the fundamental neighborhoods which spiral around $\pm\sqrt{\epsilon}$.) The map transforming H to H_1 is a global diffeomorphism of $\mathbb{CP}^1$ preserving 0 and ∞: it is hence a linear map. Then $H(x,y,\epsilon)=C(\epsilon)H_1(x,y_1(x,y,\epsilon),\epsilon)$ with $C(\epsilon)\neq0$, yielding that

$$Y(x,y,\epsilon)=\frac{H(x,y,\epsilon)}{F_\epsilon(x)}=C(\epsilon)(y_1(x,y,\epsilon)-k_{\hat\epsilon}(x))(x+\sqrt{\epsilon})^{-\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}} \tag{4.12}$$

in the neighborhood of $\sqrt{\epsilon}$. Hence $Y(x,y,\epsilon)$ vanishes at $x=\sqrt{\epsilon}$ and has the desired asymptotic properties.

Near $-\sqrt{\epsilon}$ we could not find such an elegant argument. Our proof is more analytic. In this case the first integral takes small values. We must show that it remains small when divided by $F_{\hat{\epsilon}}(x)$. For that we use that $H(x+\sqrt{\epsilon})^{-\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}$ is bounded in the neighborhood of $(x,y)=(-\sqrt{\epsilon},0)$. This comes from the construction of the spherical coordinate. Indeed the first derivative of the holonomy map at $-\sqrt{\epsilon}$ is given by $\exp(-\frac{2\sqrt{\epsilon}}{1-a\sqrt{\epsilon}})$. When building the spherical coordinate the point $-\sqrt{\epsilon}$ is sent to the origin and the holonomy restricted to the sphere becomes the identity. A ramification $x\mapsto (x+\sqrt{\epsilon})^{-\frac{2\sqrt{\epsilon}}{1-a\sqrt{\epsilon}}}+\dots$ is necessary to achieve this. The rest follows by remarking that the first integral $H(x,y,\epsilon)$ vanishes along $y=g_{\hat{\epsilon}}(x)$. □

5 Complete invariant of orbital analytic classification for a generic family of vector fields unfolding a generic saddle-node

Definition 5.1 Two germs of analytic families of vector fields, $v_{\epsilon_1}(x_1,y_1)$ (resp. $w_{\epsilon_2}(x_2,y_2)$) unfolding a saddle-node at the origin for $\epsilon_1=0$ (resp. $\epsilon_2=0$) are *weakly orbitally equivalent* if there exists a germ of map $K=(h,\Phi,k)$, $(\epsilon_1,x_1,y_1)\mapsto(h(\epsilon_1),\Phi(\epsilon_1,x_1,y_1),k(\epsilon_1,x_1,y_1))$ fibered over the parameter space where

i) $h:\epsilon_1\mapsto\epsilon_2=h(\epsilon_1)$ is a germ of homeomorphism preserving the origin.

ii) There exists a representative $\Phi_{\epsilon_1}(x_1,y_1)=\Phi(\epsilon_1,x_1,y_1)$ which is an analytic diffeomorphism in (x_1,y_1) on a small neighborhood of the origin in (x_1,y_1)-space, for ϵ_1 in a sufficiently small neighborhood of the origin.

iii) There exists a representative $k_{\epsilon_1}(x_1,y_1)=k(\epsilon_1,x_1,y_1)$ depending analytically of (x_1,y_1) for (x_1,y_1) in a small neighborhood of the origin in (x_1,y_1)-space with values in $\mathbb{C}$, and non-vanishing in a neighborhood of the origin for ϵ_1 in a sufficiently small neighborhood of the origin.

iv) The change of coordinates Φ_{ϵ_1} and the scaling of time k_{ϵ_1} is an equivalence between $v_{\epsilon_1}(x_1,y_1)$ and $w_{h(\epsilon_1)}(x_2,y_2)$ over a ball of small radius $r>0$:

$$w_{h(\epsilon_1)}(\Phi_{\epsilon_1}(x_1,y_1))=k(\epsilon_1,x_1,y_1)(\Phi_{\epsilon_1})_*(v_{\epsilon_1}(x_1,y_1)). \tag{5.1}$$

Theorem 5.2 *We consider a family unfolding a generic saddle-node and its orbital prenormal form* (2.3)*. The modulus of analytic classification of the holonomy map of the unfolded vector field, namely the family* $(\psi^0_\epsilon,\psi^\infty_\epsilon)_{\hat{\epsilon}\in V}$*, together with* $a(0)$*, is a complete modulus of orbital analytic classification under weak orbital equivalence.*

Proof. We consider two families of the form (2.3) with respective coordinates (x_1,y_1) and (x_2,y_2)

$$\begin{aligned}\dot{x}_1 &= x_1^2-\epsilon_1\\ \dot{y}_1 &= (x_1^2-\epsilon_1)f_0(x_1)+y_1(1+a_1(\epsilon_1)x_1)+o(y_1),\end{aligned} \tag{5.2}$$

and

$$\begin{array}{rcl}\dot{x}_2 & = & x_2^2 - \epsilon_2 \\ \dot{y}_2 & = & (x_2^2 - \epsilon_2)g_0(x_2) + y_2(1 + a_2(\epsilon_2)x_2) + o(y_2).\end{array} \tag{5.3}$$

such that $a_1(0) = a_2(0)$. As the parameter is an analytic invariant we can suppose that $\epsilon_1 = \epsilon_2 = \epsilon$. Moreover we must show that there exists a change of coordinate transforming the first system for a value of ϵ to the second system for the same value of ϵ. By choosing the coordinates appropriately on the spheres we can suppose that the two families have the same invariants $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})_{\hat{\epsilon}\in V}$ valid in a common sectorial neighborhood V of $\epsilon = 0$. We fix $\hat{\epsilon} \in V$: this allows not to write the dependence in $\hat{\epsilon}$. We construct changes of coordinates $\phi_1(x_1, y_1) = (x_1, Y_1(x_1, y_1, \epsilon))$ (resp. $\phi_2(x_2, y_2) = (x_2, Y_2(x_2, y_2, \epsilon))$) (depending on $\hat{\epsilon}$) transforming (5.2) (resp. (5.3)) into the family model (4.8) as in Theorem 4.7. These changes of coordinates are ramified at $\pm\sqrt{\epsilon}$. Then the change of coordinate $\Phi = \phi_2^{-1} \circ \phi_1$ is analytic in a neighborhood $U \times W$ and transforms the first system into the second. Indeed if we call L the Lavaurs map (see proof of Theorem 4.7) and we call H_j^+, Y_j^+ and ϕ_j^+ (resp. H_j^-, Y_j^- and ϕ_j^-) the branches of H_j, Y_j, ϕ_j obtained by turning around $-\sqrt{\epsilon}$ in the positive (resp. negative direction) then $H_j^+ = L \circ \psi^0_{\hat{\epsilon}}(H_j^-)$, yielding

$$Y_j^+ = \frac{1}{F(x)} L \circ \psi^0_{\hat{\epsilon}}(F(x)Y_j^-). \tag{5.4}$$

Similarly when turning around $+\sqrt{\epsilon}$. Hence $(\phi_2^+)^{-1} \circ \phi_1^+ = (\phi_2^-)^{-1} \circ \phi_1^-$, i.e. Φ is analytic and not ramified. □

An equivalent of Theorem 3.3 holds for the saddle.

Theorem 5.3 *We consider a family unfolding a generic saddle-node, its orbital prenormal form* (2.3) *and its modulus of orbital analytic classification* (ψ^0, ψ^∞) *as in Theorem* 5.2. *Let*

$$\psi^0(w) = w + \sum_{n\geq 2} b_n w^n. \tag{5.5}$$

Then $\forall p, q, k \in \mathbb{N}$ *with* $1 \leq q < p$ *there exist polynomials*

$$L^\infty_{p,q,k}(b_2, \ldots, b_{kp+1}) \tag{5.6}$$

such that if $L^\infty_{p,q,k}(b_2, \ldots, b_{kp+1}) \neq 0$ *then the saddle point* $(-\sqrt{\epsilon}, 0)$ *is non integrable of order* $\leq k$ *as soon as the ratio of its eigenvalues is of the form* $-\frac{p}{n}$ *with* n *large and* $n \equiv q \ (mod\, p)$. *If* ψ^0 *is nonlinear, then at least one of the* $L^\infty_{p,q,k}(b_2, \ldots, b_{kp+1}) \neq 0$.

Proof. The proof follows from the corresponding theorem for the holonomy map (see Theorem 8.1 of [10] and [11]): the polynomial $L_{p,q,k}$ is the k-th coefficient of the normal form of the resonant diffeomorphism $\exp(2\pi i q/p)\psi^0(w)$. □

We call this phenomenon *parametric resurgence*. Indeed the divergence of the normalizing transformation for a saddle-node coming from the nonlinearity of ψ^0 unfolds as an "incompatibility" in the system. This incompatibility is necessarily carried by the singular point of saddle type for discrete sequences of parameter values. Similarly the nonlinearity of ψ^∞ induces a parametric resurgence phenomenon at the node (see Theorem 3.3).

6 Study of an example: the Riccati equation

6.1 The general case

We study the following family of Riccati equations

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= y(1 + a(\epsilon)x) + f_0(x,\epsilon)(x^2 - \epsilon) + f_2(x,\epsilon)y^2. \end{aligned} \tag{6.1}$$

Proposition 6.1 *There exists a change of coordinate*

$$Y = \frac{y}{1 + B(\epsilon)y + C(\epsilon)(x - \sqrt{\epsilon})y} \tag{6.2}$$

depending analytically on ϵ belonging to the Riemann surface of $\sqrt{\epsilon}$ and transforming (6.1) into

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{Y} &= Y(1 + a(\epsilon)x + O(x^2 - \epsilon)) + \overline{f}_0(x)(x^2 - \epsilon) + \overline{f}_2(x)(x^2 - \epsilon)Y^2. \end{aligned} \tag{6.3}$$

A further change $X = X(x,\epsilon)$ brings the system to the form

$$\begin{aligned} \dot{X} &= X^2 - \epsilon \\ \dot{Y} &= Y(1 + \overline{a}(\epsilon)X) + \tilde{f}_0(X)(X^2 - \epsilon) + \tilde{f}_2(X)(X^2 - \epsilon)Y^2. \end{aligned} \tag{6.4}$$

Proof. We let

$$f_2(x,\epsilon) = (x^2 - \epsilon)K(x,\epsilon) + c(\epsilon)(x - \sqrt{\epsilon}) + b(\epsilon).$$

Replacing (6.2) into (6.1) and asking the transformed equation to have the form (6.3) yields to

$$\begin{cases} B(\epsilon) = \frac{b(\epsilon)}{1 + a\sqrt{\epsilon}} \\ C(\epsilon) = \frac{c(\epsilon) - aB(\epsilon)}{1 - a\sqrt{\epsilon}}. \end{cases} \tag{6.5}$$

The change in X is done using Kostov's theorem as in Section 3. □

Hence from now on we only study the particular family of Riccati equations

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= y(1 + ax) + f_0(x)(x^2 - \epsilon) + f_2(x)(x^2 - \epsilon)y^2, \end{aligned} \tag{6.6}$$

where f_0, f_2 are analytic functions of x in a neighborhood of the origin. This equation has the two singular points $P_\pm = (\pm\sqrt{\epsilon}, 0)$. These are well defined if we restrict ϵ to a sector V of (3.6). The change of coordinate $Y = \frac{1}{y}$ transforms (6.6) into

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{Y} &= -Y(1 + ax) - f_2(x)(x^2 - \epsilon) - f_0(x)(x^2 - \epsilon)Y^2, \end{aligned} \tag{6.7}$$

which has the singular points $Q_\pm$ given by $(x, Y) = (\pm\sqrt{\epsilon}, 0)$. It is this change of coordinate which allows a complete calculation of the Martinet-Ramis modulus when $\epsilon = 0$ ([8]).

There exists a neighborhood U of the origin in x-space such that, if ρ is sufficiently small, then $\pm\sqrt{\epsilon} \in U$. In the following we will put additional conditions on U.

The point P_- always has an analytic separatrix $y - g_0(x) = 0$ (depending on $\hat{\epsilon}$) when $\hat{\epsilon}$ is in the sector V. This function $g_0(x)$ is the analytic solution of the differential equation

$$(x^2-\epsilon)g_0'(x) = g_0(x)(1+ax) + (x^2-\epsilon)f_0(x) + (x^2-\epsilon)g_0^2(x)f_2(x), \tag{6.8}$$

satisfying $g_0(-\sqrt{\epsilon}) = 0$. From Theorem 3.2 the function $g_0(x)$ is defined for all $\hat{\epsilon} \in V$ as a multivalued function over the domain U containing $\pm\sqrt{\epsilon}$, independent of ϵ, and ramified at $\sqrt{\epsilon}$. For $\epsilon = 0$ the graph of this function is the center manifold.

Similarly the point Q_+ of system (6.7) always has an analytic separatrix of the form $Y - g_2(x) = 0$ where $g_2(x)$ is a solution of the differential equation

$$(x^2-\epsilon)g_2'(x) + (1+ax)g_2(x) + (x^2-\epsilon)f_2(x) + (x^2-\epsilon)f_0(x)g_2^2(x) = 0, \tag{6.9}$$

with $g_2(\sqrt{\epsilon}) = 0$. As for g_0 the function $g_2(x)$ can be defined for all $\hat{\epsilon} \in V$ as a function on U ramified at $-\sqrt{\epsilon}$.

We construct a first integral of this equation by means of generalized Darboux factors and cofactors (see for instance [2]). Recall that a generalized Darboux factor is a function $G(x,y)$ such that

$$\dot{G}(x,y) = \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\dot{y} = G(x,y)K(x,y). \tag{6.10}$$

The function $K(x,y)$ is called the cofactor of $G(x,y)$. On purpose we do not define the classes of functions to which belong $F(x,y)$ and $K(x,y)$. Indeed the differential equation (6.8) (resp. (6.9)) satisfied by $g_0(x)$ (resp. $g_2(x)$) allows to consider $y - g_0(x)$ (resp. $1 - yg_2(x)$) as a Darboux factor with cofactor given in (6.11) below.

The system (6.6) has the following generalized Darboux factors and cofactors (we simply write a for $a(\epsilon)$):

$$\begin{array}{ll}
F_1(x) = x - \sqrt{\epsilon} & K_1(x,y) = x + \sqrt{\epsilon} \\
F_2(x) = x + \sqrt{\epsilon} & K_2(x,y) = x - \sqrt{\epsilon} \\
F_3(x,y) = y - g_0(x) & K_3(x,y) = (x^2-\epsilon)f_2(x)y + 1 + ax \\
 & \qquad +(x^2-\epsilon)f_2(x)g_0(x) \\
F_4(x,y) = 1 - g_2(x)y & K_4(x,y) = (x^2-\epsilon)f_2(x)y \\
 & \qquad -(x^2-\epsilon)f_0(x)g_2(x) \\
F_5(x,y) = \exp\left(-\int_{-\sqrt{\epsilon}}^{x} f_2(\xi)g_0(\xi)d\xi\right) & K_5(x) = -(x^2-\epsilon)f_2(x)g_0(x) \\
F_6(x,y) = \exp\left(-\int_{\sqrt{\epsilon}}^{x} f_0(\xi)g_2(\xi)d\xi\right) & K_6(x) = -(x^2-\epsilon)f_0(x)g_2(x)
\end{array} \tag{6.11}$$

As there exists a linear combination $\sum_{i=1}^{6} \beta_i K_i = 0$, this yields a first integral $H_\epsilon(x,y) = \prod_{i=1}^{6} F_i^{\beta_i}$, i.e.

$$H_\epsilon(x,y) = (x^2-\epsilon)^{-\frac{a}{2}} \left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}} F_5(x)F_6(x)\frac{y-g_0(x)}{1-yg_2(x)}. \tag{6.12}$$

where $F_5(x) = 1 + O(x+\sqrt{\epsilon})$, $F_6(x) = 1 + O(x-\sqrt{\epsilon})$.

For each ϵ the first integral is defined on a domain $U \times W$ where W is a small disk in y-space. The first integral is ramified at $\pm\sqrt{\epsilon}$ and we can consider it as a univalued function defined on $\hat{U} \times W$ where $\hat{U}$ is a domain as in Figure 5a).

When turning around $\pm\sqrt{\epsilon}$ this yields two integrals $H_{1,\pm,\epsilon}$ and $H_{2,\pm,\epsilon}$ depending whether we turn around $\pm\sqrt{\epsilon}$ in the positive (resp. negative) direction.

Proposition 6.2 *(1)* $H_{2,+,\epsilon} = A(\epsilon)H_{1,+,\epsilon} + B(\epsilon)$,

(2) $H_{2,-,\epsilon} = \frac{C(\epsilon)H1,-,\epsilon}{1+D(\epsilon)H_{1,-,\epsilon}}$,
where $A(\epsilon), C(\epsilon) \neq 0$ *and* $B(\epsilon), D(\epsilon)$ *depend continuously on* $\hat{\epsilon} \in V$.

PROOF. The functions $g_0(x)$ and $F_5(x)$ are ramified at $x = \sqrt{\epsilon}$ while the functions $g_2(x)$ and $F_6(x)$ are ramified at $-\sqrt{\epsilon}$. Also the map

$$F(x) = (x^2 - \epsilon)^{-\frac{a}{2}} \left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}} \tag{6.13}$$

is ramified at the two points $\pm\sqrt{\epsilon}$. Let us call $\overline{g}_0, \overline{F}_5, \overline{F}$ (resp. $\hat{g}_0, \hat{F}_5, \hat{F}$) the extensions of g_0, F_5, F when turning around $\sqrt{\epsilon}$ in the positive (resp. negative) direction. Then

$$\begin{array}{rcl} H_{1,+,\epsilon} & = & \overline{F}(x)\overline{F}_5(x)F_6(x)\frac{y-\overline{g}_0(x)}{1-yg_2(x)}, \\ H_{2,+,\epsilon} & = & \hat{F}(x)\hat{F}_5(x)F_6(x)\frac{y-\hat{g}_0(x)}{1-yg_2(x)}. \end{array} \tag{6.14}$$

We eliminate y between the two equations of (6.14), yielding

$$H_{2,+,\epsilon} = \frac{\hat{F}(x)\hat{F}_5(x)(1-g_2(x)\hat{g}_0(x))}{\overline{F}(x)\overline{F}_5(x)(1-g_2(x)\overline{g}_0(x))}H_{1,+,\epsilon} + \frac{\hat{F}(x)\hat{F}_5(x)F_6(x)(\overline{g}_0(x)-\hat{g}_0(x))}{1-g_2(x)\overline{g}_0(x)}. \tag{6.15}$$

Similarly let us call $\tilde{g}_2, \tilde{F}_6, \tilde{F}$ (resp. $\check{g}_2, \check{F}_6, \check{F}$) the extensions of g_2, F_6, F when turning around $-\sqrt{\epsilon}$ in the positive (resp. negative) direction. Then

$$\begin{array}{rcl} H_{1,-,\epsilon} & = & \tilde{F}(x)F_5(x)\tilde{F}_6(x)\frac{y-g_0(x)}{1-y\tilde{g}_2(x)} \\ H_{2,-,\epsilon} & = & \check{F}(x)F_5(x)\check{F}_6(x)\frac{y-g_0(x)}{1-y\check{g}_2(x)}. \end{array} \tag{6.16}$$

We eliminate y between the two equations of (6.16), yielding

$$H_{2,-,\epsilon} = \frac{\check{F}(x)F_5(x)\check{F}_6(x)(1-\tilde{g}_2(x)g_0(x))H_{1,-,\epsilon}}{(\tilde{g}_2(x)-\check{g}_2(x))H_{1,-,\epsilon} + \tilde{F}(x)F_5(x)\tilde{F}_6(x)(1-g_0(x)\check{g}_2(x))}. \tag{6.17}$$

□

6.2 The particular case of the linear equation

We limit ourselves to the case of $f_2 \equiv 0$ in (6.1). In this case ψ_ϵ^0 is linear and $\psi_\epsilon^\infty(w) = w + C(\epsilon)$ for adequate choices of coordinates on the spheres. We calculate explicitly $C(\epsilon)$ in this case. Note that the general case can be reduced to this particular case when the function $g_2(x,\epsilon)$ is analytic.

Theorem 6.3 *i) Let* $f_0(x) = \sum_{n\geq 0} b_n(\epsilon)(x+\sqrt{\epsilon})^n$. *Then the modulus has the form* $(\psi_\epsilon^0, \psi_\epsilon^\infty)$ *with* ψ_ϵ^0 *linear and* $\psi_\epsilon^\infty(w) = w + C(\epsilon)$ *where*

$$C(\epsilon) = -2i\pi \frac{\Gamma(\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}})(2\sqrt{\epsilon})^{-a}}{\Gamma(\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}})} \sum_{n\geq 0} \frac{b_n(\epsilon)}{\Gamma(n-a+2)} \prod_{j=0}^{n} (2j\sqrt{\epsilon} + 1 - a\sqrt{\epsilon}). \tag{6.18}$$

The function $C(\epsilon)$ is analytic in $\sqrt{\epsilon}$ near $\epsilon=0$ in the particular case $a(\epsilon)\equiv m$ with $m\in\mathbb{Z}$. Otherwise it is a quotient of two functions in $\sqrt{\epsilon}$, analytic for $\epsilon\neq 0$ and having each an essential singularity at $\epsilon=0$. For ϵ going to 0 in the sectorial neighborhood V the limit value is

$$C(0)=-2\pi i\sum_{n\geq 0}\frac{b_n(0)}{\Gamma(n-a+2)}. \tag{6.19}$$

ii) The modulus space is isomorphic to $\mathcal{H}_0^2$ where $\mathcal{H}_0$ is the set of germs of holomorphic functions in $\sqrt{\epsilon}$ at the origin. Given $(a(\epsilon),d(\epsilon))\in\mathcal{H}_0^2$ the modulus is given by

$$\psi_\epsilon^0(w)=e^{-2\pi i a(\epsilon)}w,\qquad \psi_\epsilon^\infty(w)=w-2i\pi\frac{\Gamma(\frac{1-a(\epsilon)\sqrt{\epsilon}}{2\sqrt{\epsilon}})(2\sqrt{\epsilon})^{-a(\epsilon)}}{\Gamma(\frac{1+a(\epsilon)\sqrt{\epsilon}}{2\sqrt{\epsilon}})}d(\epsilon).$$

iii) Any family (6.1) with $f_2\equiv 0$ is weakly orbitally equivalent to a family of the form

$$\begin{array}{rcl}\dot{x}&=&x^2-\epsilon\\ \dot{y}&=&y(1+a(\epsilon)x)+c_N(\epsilon)x^N\end{array} \tag{6.20}$$

with $N>|a(\epsilon)|$.

Proof. We know that the first integral has the form

$$H_\epsilon(x,y)=(y-g_0(x))(x^2-\epsilon)^{-\frac{a}{2}}\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}=(y-g_0(x))F(x), \tag{6.21}$$

where $F(x)$ is given in (6.13). So we need only understand the behavior of the function $y=g_0(x)$ which is the analytic separatrix of $-\sqrt{\epsilon}$ (when this point is not a node). It has the form

$$g_0(x)=(x+\sqrt{\epsilon})^{-\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}(x-\sqrt{\epsilon})^{\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}\int_{-\sqrt{\epsilon}}^x f_0(\zeta)(\zeta+\sqrt{\epsilon})^{\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}(\zeta-\sqrt{\epsilon})^{-\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}d\zeta. \tag{6.22}$$

To calculate the integral part we make the change of coordinate $\xi=\frac{\zeta+\sqrt{\epsilon}}{2\sqrt{\epsilon}}$ in the integral. Except for special values of ϵ for which $\sqrt{\epsilon}$ is a resonant node the function $g_0(x)$ generically has the form

$$g_0(x)=(x+\sqrt{\epsilon})^{-\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}(x-\sqrt{\epsilon})^{\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}(A(\epsilon)+O(x-\sqrt{\epsilon}))$$

in the neighborhood of $\sqrt{\epsilon}$. We calculate $A(\epsilon)$. It is given by

$$\begin{array}{rcl}A(\epsilon)&=&\sum_{n\geq 0}b_n(\epsilon)(2\sqrt{\epsilon})^{n-a+1}e^{-\pi i\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}\int_0^1\xi^{n+\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}(1-\xi)^{-\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}d\xi\\ &=&\sum_{n\geq 0}b_n(\epsilon)(2\sqrt{\epsilon})^{n-a+1}e^{-\pi i\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}\frac{\Gamma(n+1+\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}})\Gamma(1-\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}})}{\Gamma(n-a+2)}.\end{array} \tag{6.23}$$

We must now see how the first integral H of (6.21) ramifies when we turn around $\sqrt{\epsilon}$. Let us call H_1 the value of h when we make one turn around $\sqrt{\epsilon}$. And let us treat the case when the node is not resonant as it is sufficient because of the analytic character of $C(\epsilon)$. (What happens when the node is resonant will be described below). Locally near $\sqrt{\epsilon}$ we have that $y-g_0(x)=y-h(x)-A(\epsilon)F^{-1}(x)$ where $h(x)$ is the analytic solution of (6.8) (which is here a linear equation) in the neighborhood of $\sqrt{\epsilon}$ and $-A(\epsilon)F^{-1}(x)$ is a solution of the homogeneous system. Hence the first integral has the form $H=(y-h(x))F(x)-A(\epsilon)$. When we make one turn around $\sqrt{\epsilon}$ it becomes $H_1=B(\epsilon)(y-h(x))F(x)-A(\epsilon)$, with

$$B(\epsilon)=e^{-2\pi i\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}.$$

Hence

$$H_1=B(\epsilon)(H+A(\epsilon))-A(\epsilon)=B(\epsilon)\left(H+A(\epsilon)\frac{B(\epsilon)-1}{B(\epsilon)}\right)=B(\epsilon)(H+C(\epsilon)),$$

with $C(\epsilon)=A(\epsilon)\frac{B(\epsilon)-1}{B(\epsilon)}$. We finally get

$$\begin{aligned} C(\epsilon) &= -2i\textstyle\sum_{n\geq 0} b_n(\epsilon)(2\sqrt{\epsilon})^{n-a+1}\sin\left(\pi\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}\right)\frac{\Gamma(n+1+\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}})\Gamma(1-\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}})}{\Gamma(n-a+2)} \\ &= -2\pi i\textstyle\sum_{n\geq 0}\frac{b_n(\epsilon)(2\sqrt{\epsilon})^{n-a+1}}{\Gamma(n-a+2)}\frac{\Gamma(\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}})}{\Gamma(\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}})}\prod_{j=0}^{n}\left(j+\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}\right), \end{aligned} \tag{6.24}$$

using $\Gamma(z+1)=z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, from which (6.18) follows.

In the particular case $a(\epsilon)\equiv m$, $m\in\mathbb{Z}$ then the formula takes the simple form

$$C(\epsilon)=-2\pi i e(\epsilon)\sum_{n\geq 0}\frac{b_n(\epsilon)}{\Gamma(n-a+2)}\prod_{j=0}^{n}\left(2j\sqrt{\epsilon}+1-m\sqrt{\epsilon}\right), \tag{6.25}$$

where

$$e(\epsilon)=\begin{cases}\prod_{j=1}^{-m}\left(1+2\sqrt{\epsilon}(-\frac{m}{2}-j)\right) & m<0\\ \prod_{j=1}^{m}\left(1+2\sqrt{\epsilon}(\frac{m}{2}-j)\right)^{-1} & m>0\\ 1 & m=0.\end{cases}$$

In particular we see that $C(\epsilon)$ is analytic in $\sqrt{\epsilon}$ near $\epsilon=0$ when $a(\epsilon)\equiv m$, $m\in\mathbb{Z}$.

In the limit when ϵ is very small we use

$$\Gamma(\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}})\sim\left(\frac{1}{2\sqrt{\epsilon}}\right)^{-a}\Gamma(\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}) \tag{6.26}$$

to get the limit (6.19), which is the formula of Martinet-Ramis [8].

The function $1/\Gamma$ is entire with an essential singularity at infinity. The series in (6.18) converges, yielding that $C(\epsilon)$ is a uniform function multiplied by a quotient of two essential singularities at $\sqrt{\epsilon}=0$. It is accumulated by a sequence of zeroes (resp. a sequence of poles) located at the values of ϵ where $\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}$ (resp. $\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}$) is a negative integer ≤ -1 (resp. ≤ -2). The zeroes and poles annihilate each other when $a(\epsilon)=0$.

Part iii) follows from Theorem 5.2 and taking

$$c_N(\epsilon)=\frac{\Gamma(N-a+2)}{\prod_{j=0}^{N}(2j\sqrt{\epsilon}+1-a\sqrt{\epsilon})}\sum_{n\geq 0}\frac{b_n(\epsilon)}{\Gamma(n-a+2)}\prod_{j=0}^{n}(2j\sqrt{\epsilon}+1-a\sqrt{\epsilon}). \tag{6.27}$$

□

Remark: The phenomemon near a resonant node with ratio of eigenvalues n is interesting. The constant $B(\epsilon)-1$ vanishes as soon as the node is resonant since indeed no solutions are ramified in the case of a linearizable resonant node. To understand what happens at a resonant node, i.e. when $\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}=n$, we let $\alpha=\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}-n$ and look at what happens when $\alpha\to 0$. As the system is close to a resonant node, i.e. of the form $x\frac{\partial}{\partial x}+[(n+\alpha)y+\beta x^n)]\frac{\partial}{\partial y}$, if $\beta\neq 0$ then the function $h(x)$ (the analytic solution at $x=\sqrt{\epsilon}$) contains a term in $(x-\sqrt{\epsilon})^n$ with a very large coefficient $A(\epsilon)$. If $C(\epsilon)\neq 0$ then $g_0(x)$ contains a ramified coefficient of the form $-A(\epsilon)(x-\sqrt{\epsilon})^n$. The sum of these two terms is of the form $\alpha A(\epsilon)(x-\sqrt{\epsilon})^n\omega(x-\sqrt{\epsilon},\alpha)$, where

$$\omega(x-\sqrt{\epsilon},\alpha)=\begin{cases}\frac{(x-\sqrt{\epsilon})^{-\alpha}-1}{\alpha} & \alpha\neq 0\\ -\ln(x-\sqrt{\epsilon}) & \alpha=0.\end{cases}$$

We know that when $C(0)\neq 0$, then $C(\epsilon)\neq 0$ for ϵ sufficiently small, yielding that $A(\epsilon)$ becomes infinite at all resonant nodes. However as $A(\epsilon)$ has a simple pole when $\alpha=0$ the limit of $\alpha A(\epsilon)$ exists when $\alpha\to 0$.

7 Directions for further research

We mention two natural directions for further research:

(1) The first is to identify precisely the modulus space for analytic families of vector fields unfolding a saddle-node. This amounts to identify precisely which families $(\psi^0_{\hat{\epsilon}},\psi^\infty_{\hat{\epsilon}})$ are realizable as the modulus of an analytic unfolding of a saddle-node. The corresponding problem is open in the parabolic case. The difficulty comes from the unknown behaviour in $\hat{\epsilon}$ of the $\psi^{0,\infty}_{\hat{\epsilon}}$ at $\hat{\epsilon}=0$. We conjecture that we have more than continuity in $\sqrt{\epsilon}$. The linear example shows that analytic families in $\sqrt{\epsilon}$ may not be realizable when $a(\epsilon)\not\equiv m\in\mathbb{Z}$. In the particular case $a(\epsilon)\equiv m\in\mathbb{Z}$ we do not even know if families $(\psi^0_\epsilon,\psi^\infty_\epsilon)$ analytic in $\sqrt{\epsilon}$ are realizable as the modulus of an analytic unfolding of a germ of diffeomorphism with a parabolic point, although we believe this is the case.

(2) The second is to generalize the previous results for a saddle-node of arbitrary codimension.

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