KAM FOR THE NON-LINEAR SCHRÖDINGER EQUATION – A NOT SO SHORT PRESENTATION

L. H. ELIASSON AND S. B. KUKSIN

1. Introduction

1.1. **The non-linear Schrödinger equation.** We consider the d-dimensional nonlinear Schrödinger equation

$$-i\dot{u} = -\Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x)$$

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution potential $V : \mathbb{T}^d \to \mathbb{C}$ have real Fourier coefficients $\hat{V}(a),\ a \in \mathbb{Z}^d$, and we shall suppose it is analytic. (This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu.) F is an analytic function in $\Re u, \Im u$ and x. When $F(x, u, \bar{u}) = (u\bar{u})^2$ this is the cubic Schrödinger equation.

For $\varepsilon = 0$ the equation is linear and has time–quasi-periodic solutions

$$u(t, x) = \sum_{a \in \mathcal{A}} \hat{u}(a) e^{i(|a|^2 + \hat{V}(a))t} e^{i<a,x>} \quad (0 < |\hat{u}(a)|),$$

where $\mathcal{A}$ is any finite subset of $\mathbb{Z}^d$. For $\varepsilon \neq 0$ we have

If $|\varepsilon|$ is sufficiently small, then there is a large subset U' of U such that for all $\omega \in U'$ the solution u persists as a time–quasi-periodic solution which has all Lyapounov exponents equal to zero and whose linearized equation is reducible to constant coefficients.

This is a not so short presentation of the basic ideas behind this result. A detailed proof is given in [EK06].

1.2. **An ∞-dimensional Hamiltonian system.** We write

$$\begin{cases} u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{i<a,x>} \\ \overline{u(x)} = \sum_{a \in \mathbb{Z}^d} v_a e^{i<-a,x>} \quad (v_a = \bar{u}_a). \end{cases}$$

In the symplectic space $\{(u_a, v_a) : a \in \mathbb{Z}^d\} = \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}$,

$$i \sum_{a \in \mathbb{Z}^d} du_a \wedge dv_a,$$

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the equation becomes a Hamiltonian system

$$\begin{cases} \dot{u}_a = & i\frac{\partial}{\partial v_a}(h+\varepsilon f) \\ \dot{v}_a = & -i\frac{\partial}{\partial u_a}(h+\varepsilon f) \end{cases}$$

with an integrable part

$$h(u,v)=\sum_{a\in\mathbb{Z}^d}(|a|^2+\hat{V}(a))u_a v_a$$

plus a perturbation

$$\varepsilon f(u,v)=\varepsilon\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}F(x,u(x),\overline{u(x)})dx.$$

The second derivatives of f have a *Töplitz invariance* :

$$\frac{\partial^2 f}{\partial u_{a+c}\partial v_{b+c}}=\frac{\partial^2 f}{\partial u_a\partial v_b}$$

and

$$\frac{\partial^2 f}{\partial u_{a+c}\partial u_{b-c}}=\frac{\partial^2 f}{\partial u_a\partial u_b}$$

(and similar for the second derivatives with respect to v_a, v_b), for any $c\in\mathbb{Z}^d$.

This is easy to see for the cubic Schrödinger where

$$f(u,v)=\sum_{a+b-c-d=0}u_a u_b v_c v_d. \tag{1}$$

For example

$$\frac{\partial^2 f}{\partial u_a\partial u_b}=\sum_{c+d=a+b}v_c v_d$$

which clearly have this invariance.

The non-linear Schrödinger is a *real* Hamiltonian system. Indeed if we let

$$\zeta_a=\begin{pmatrix}\xi_a\\ \eta_a\end{pmatrix}=C\begin{pmatrix}u_a\\ v_a\end{pmatrix},$$

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}, \tag{2}$$

then, in the symplectic space $\{(\xi_a,\eta)=: a\in\mathbb{Z}^d\}=\mathbb{C}^{\mathbb{Z}^d}\times\mathbb{C}^{\mathbb{Z}^d}$,

$$\sum_{a\in\mathbb{Z}^d}d\xi_a\wedge d\eta_a,$$

the equation becomes

$$\begin{cases} \dot{\xi}_a = -\frac{\partial}{\partial \eta_a}(h+\varepsilon f) \\ \dot{\eta}_a = \quad \frac{\partial}{\partial \xi_a}(h+\varepsilon f) \end{cases} = J\frac{\partial}{\zeta_a}(h+\varepsilon f), \qquad a \in \mathbb{Z}^d,$$

with the integrable part

$$h(\xi,\eta) = \frac{1}{2}\sum_{a\in\mathbb{Z}^d}(|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2)$$

plus the perturbation $\varepsilon f(\xi,\eta)$ which is real, because F is a real function in $\Re u$ and $\Im u$.

The Töplitz-invariance of the second derivatives can of course be translated to these coordinates but the description is more complicated (see Section 7.1).

1.3. **The topology.** Let $\mathcal{L}$ be a subset of $\mathbb{Z}^d$. The space

$$l^2_\gamma(\mathcal{L},\mathbb{R}), \quad \gamma \geq 0,$$

is the set of sequences of real numbers $\xi = \{\xi_a : a \in \mathcal{L}\}$, such that

$$\|\xi\|_\gamma = \sqrt{\sum_{a\in\mathcal{L}} |\xi_a|^2 \langle a\rangle^{2m_*} e^{2\gamma|a|}} < \infty \qquad \langle a\rangle = \max(|a|,1).$$

There is a natural identification of $l^2_\gamma(\mathcal{L},\mathbb{R}) \times l^2_\gamma(\mathcal{L},\mathbb{R})$, whose elements are (ξ,η), with $l^2_\gamma(\mathcal{L},\mathbb{R}^2)$, whose elements are $\{\xi_a,\eta_a) : a \in \mathcal{L}\}$, and we will not distinguish between them.

We shall assume that $m_* > \frac{d}{2}$. Then, in the phase space $l^2_0(\mathbb{Z}^d,\mathbb{R}^2)$, our Hamiltonian $h+\varepsilon f$ is analytic (in some domain $\mathcal{O}$). To see that f is analytic, consider for example the cubic Schrödinger in the complex variables (1). Using the estimate

$$\sum_a |u_a| \leq \sqrt{\sum_a (\langle a\rangle)^{-2m_*}}\, \|u\|_0\,,$$

we have

$$|f(u,v)| \leq \|u\|_0^2\, \|v\|_0^2\,,$$

and it follows easily that f is analytic.

Since the phase space is a Hilbert space, its first differential

$$l^2_0(\mathcal{L},\mathbb{R}^2) \ni \hat{\zeta} \mapsto <\hat{\zeta}, \partial_\zeta f(\zeta)>$$

defines a unique vector $\partial_\zeta f(\zeta)$ in $l^2_0(\mathcal{L},\mathbb{R}^2)$, its *gradient*, and its second differential

$$l^2_0(\mathcal{L},\mathbb{R}^2) \ni \hat{\zeta} \mapsto \frac{1}{2}<\hat{\zeta}, \partial^2_\zeta f(\zeta)\hat{\zeta}>$$

defines a unique matrix $\partial^2_\zeta f(\zeta)$ $\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{R})$, its *Hessian*, which is symmetric, i.e.

$$^t\!\Big(\frac{\partial^2 f}{\partial z_a\partial z_b}(\zeta)\Big)=\frac{\partial^2 f}{\partial z_b\partial z_a}(\zeta).$$

(Here $<\cdot>$ is the scalar product of the phase space.)

For $\zeta\in\mathcal{O}\cap l^2_\gamma(\mathbb{Z}^d,\mathbb{R}^2)$, $\gamma>0$, the gradient and the Hessian verifies certain properties of *exponential decay.* These properties are most easily seen in the complex variables (u,v) – consider for example the cubic Schrödinger (1). The first derivatives of f verify

$$|\frac{\partial f}{\partial u_a}|\le \text{cte.}\,\|u\|_\gamma\,\|v\|^2_\gamma\,e^{-\gamma|a|}.$$

(and similar for the derivative with respect to v_a). The second derivatives verify

$$|\frac{\partial^2 f}{\partial u_a\partial v_c}|\le \text{cte.}\,\|u\|_\gamma\,\|v\|_\gamma\,e^{-\gamma|a-c|},$$

and

$$|\frac{\partial^2 f}{\partial u_a\partial u_b}|\le \text{cte.}\,\|v\|^2_\gamma\,e^{-\gamma|a+b|}$$

(and similar for the second derivative with respect to v_c,v_d).

The exponential decay of the second derivatives Tcan of course be translated to the real coordinates (ξ,η) but the description is more complicated (see Section 7.1).

1.4. **Action-angle variables.** Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$ and fix

$$0<p_a,\quad a\in\mathcal{A}.$$

The $(\#\mathcal{A})$-dimensional torus

$$\begin{array}{ll}\frac12(\xi_a^2+\eta_a^2)=p_a & a\in\mathcal{A}\\ \xi_a=\eta_a=0 & a\in\mathcal{L}=\mathbb{Z}^d\setminus\mathcal{A},\end{array}$$

is invariant for the Hamiltonian flow when $\varepsilon=0$. In the symplectic subspace $\mathbb{R}^{\mathcal{A}}\times\mathbb{R}^{\mathcal{A}}$ we introduce, in a neighborhood of this torus, action-angle variables (r_a,φ_a), $a\in\mathcal{A}$,

$$\begin{array}{l}\xi_a=\sqrt{2(p_a+r_a)}\cos(\varphi_a)\\ \eta_a=\sqrt{2(p_a+r_a)}\sin(\varphi_a).\end{array}$$

In these coordinates the Hamiltonian equations becomes

$$\left\{\begin{array}{ll}\zeta_a=J\frac{\partial}{\partial\zeta_a}(h+\varepsilon f) & a\in\mathcal{L}\\ \dot r_a=-\frac{\partial}{\partial\varphi_a}(h+\varepsilon f) & \\ \dot\varphi_a=\ \frac{\partial}{\partial r_a}(h+\varepsilon f) & a\in\mathcal{A}\end{array}\right.$$

with the integrable part $h(\xi,\eta,r) =$

$$\sum_{a\in\mathcal{A}}\omega_a r_a + \frac{1}{2}\sum_{a\in\mathcal{L}}\Omega_a(\xi_a^2+\eta_a^2)$$

(modulo a constant), where

$$\omega_a = |a|^2 + \hat{V}(a), \quad a\in\mathcal{A},$$

are the *basic frequencies*, and

$$\Omega_a = |a|^2 + \hat{V}(a), \quad a\in\mathcal{L},$$

are the *normal frequencies* (of the invariant torus). The perturbation $\varepsilon f(\xi,\eta,r,\varphi)$ will be a function of all variables (under the assumption, of course, that the torus lies in the domain of F).

Since $h+\varepsilon f$ is analytic on some domain in (some domain in) the phase space $l_0^2(\mathcal{L},\mathbb{R}^2)\times\mathbb{R}^{\mathcal{A}}\times\mathbb{T}^{\mathcal{A}}$, it extends to a holomorphic function on a complex domain

$$\mathcal{O}^0(\sigma,\mu,\rho) = \begin{cases} \|\zeta\|_0 = \sqrt{\|\xi\|_0^2+\|\eta\|_0^2} < \sigma \\ |r| < \mu \\ |\Im\varphi| < \rho. \end{cases}$$

1.5. **Statement of the result.** The Hamiltonian $h+\varepsilon f$ is a standard form for the perturbation theory of lower-dimensional (isotropic) tori with one exception: it is strongly degenerate. We therefore need external parameters to control the basic frequencies and the simplest choice is to let the basic frequencies (i.e. the potential itself) be our free parameters. The parameters will belong to a set

$$U \subset \{\omega\in\mathbb{R}^{\mathcal{A}} : |\omega| \le C_1\}. \tag{3}$$

The potential V will be analytic and

$$|\hat{V}(a)| \le C_2 e^{-C_3|a|},\ C_3>0,\ \forall a\in\mathcal{L}. \tag{4}$$

The normal frequencies will be assumed to verify

$$\begin{cases} |\Omega_a| \ge C_4 > 0 & \\ |\Omega_a+\Omega_b| \ge C_4 & \\ |\Omega_a-\Omega_b| \ge C_4 & |a|\ne|b| \end{cases} \qquad \forall\, a,b\in\mathcal{L}. \tag{5}$$

This is fulfilled, for example, if V is small and $\mathcal{A}\ni 0$, or if V is arbitrary and $\mathcal{A}$ is sufficiently large.

Theorem A. *Under the above assumptions, for ε sufficiently small there exist a subset $U' \subset U$, which is large in the sense that*

$$\text{Leb}\,(U \setminus U') \leq \text{cte}.\varepsilon^{exp}\,,$$

and for each $\omega \in U'$, a real analytic symplectic diffeomorphism Φ

$$\mathcal{O}^0(\frac{\sigma}{2}, \frac{\mu}{2}, \frac{\rho}{2}) \to \mathcal{O}^0(\sigma, \mu, \rho)$$

and a vector $\omega' = \omega'(\omega)$ such that $(h_{\omega'} + \varepsilon f) \circ \Phi$ equals (modulo a constant)

$$<\omega, r> + \frac{1}{2}<\zeta, A(\omega)\zeta> + \varepsilon g\,,$$

where

(i)

$$g \in \mathcal{O}(|r|^2\,, |r|\,\|\zeta\|_0\,, \|\zeta\|_0^3),$$

(ii) *the symmetric matrix $A(\omega)$ has the form*

$$\begin{pmatrix} \Omega_1(\omega) & \Omega_2(\omega) \\ {}^t\Omega_2(\omega) & \Omega_1(\omega) \end{pmatrix}$$

with $\Omega_1 + i\Omega_2$ Hermitian and block-diagonal, with finite-dimensional blocks.

Moreover,

(iii) *$\Phi = (\Phi_\zeta, \Phi_r, \Phi_\varphi)$ verifies, for all $(\zeta, \varphi, r) \in \mathcal{O}^0(\frac{\sigma}{2}, \frac{\mu}{2}, \frac{\rho}{2})$,*

$$\|\Phi_\zeta - \zeta\|_0 + |\Phi_r - \rho| + |\Phi_\varphi - \varphi| \leq \beta\varepsilon,$$

(iv) *the mapping $\omega \mapsto \omega'(\omega)$ verifies*

$$|\omega' - \text{id}|_{\text{Lip}(U')} \leq \beta\varepsilon.$$

β is a constant that depends on the dimensions $d, \#\mathcal{A}, m_*$, on the constants $C_1, \ldots, C_4$ and on V and F.

It follows from this theorem that $\Phi(\{0\} \times \{0\} \times \mathbb{T}^{\mathcal{A}})$ is a KAM-torus for the Hamiltonian system of $h + \varepsilon f$, and it implies the result mentioned in Section 1.1. We discuss this notion and its consequences in the next section.

Theorem A, as well as a more generalized version, is proven in [EK06].

1.6. **Notations.** $<,>$ is the standard scalar product in $\mathbb{R}^d$. $\| \ \|$ is an operator-norm or l^2-norm. $| \ |$ will in general denote a supremum norm, with a notable exception: for a lattice vector $a \in \mathbb{Z}^d$ we use $|a|$ for the l^2-norm.

$\mathcal{A}$ is a finite subset of $\mathbb{Z}^d$, and $\mathcal{L}$ is its complement. A matrix on $\mathcal{L}$ is just a mapping $A : \mathcal{L} \times \mathcal{L} \to \ \mathbb{C} \quad \text{or} \quad gl(2, \mathbb{C})$. Its components will be denoted A_a^b. If A_1, A_2, A_3, A_4 are scalar-valued matrices on $\mathcal{L}$, then we identify

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

with a $gl(2, \mathbb{C})$-valued matrix through

$$A_a^b = \begin{pmatrix} (A_1)_a^b & (A_2)_a^b \\ (A_3)_a^b & (A_4)_a^b \end{pmatrix}.$$

The dimension d will be fixed and m_* will be a fixed constant $> \frac{d}{2}$.

$\lesssim$ means $\leq$ modulo a multiplicative constant that only, unless otherwise specified, depends on d, m_* and $\#\mathcal{A}$.

The points in the lattice $\mathbb{Z}^d$ will be denoted $a, b, c, \ldots$. Also d will sometimes be used, without confusion we hope.

Greek letter $\alpha, \beta, \ldots$ will mostly be used for bounds. Exceptions are φ which will denote an element in the torus – an angle – and ω, Ω.

For two subsets X and Y of a metric space,

$$\text{dist}(X, Y) = \inf_{x \in X, y \in Y} d(x, y).$$

(This is not a metric.) X_ε is the ε-neighborhood of X, i.e.

$$\{y : \text{dist}(y, X) < \varepsilon\}.$$

Let $B_\varepsilon(x)$ be the ball $\{y : d(x, y) < \varepsilon\}$. Then X_ε is the union, over $x \in X$, of all $B_\varepsilon(x)$.

If X and Y are subsets of $\mathbb{R}^d$ or $\mathbb{Z}^d$ we let

$$X - Y = \{x - y : x \in X, \ y \in Y\}$$

– not to be confused with the set theoretical difference $X \setminus Y$.

2. KAM-tori

2.1. **KAM-tori.** A *KAM-torus* of a Hamiltonian system in $\mathbb{R}^{2\mathcal{L}} \times \mathbb{R}^{\mathcal{A}} \times \mathbb{T}^{\mathcal{A}}$ is a finite-dimensional torus with three properties:

(i) *invariance* – it is invariant under the Hamiltonian flow;

(ii) *linearity* – the flow on the torus is conjugate to a linear flow $\varphi \mapsto \varphi + t\omega$;

(iii) *reducibility* – the linearized equations (the "variational equations") on the torus are conjugate to a constant coefficient system of the form

$$\begin{cases} \frac{d\hat{\zeta}}{dt} = JA\hat{\zeta} \\ \frac{d\hat{r}}{dt} = 0 \\ \frac{d\hat{\varphi}}{dt} = \beta\hat{r} \end{cases}$$

and JA has a pure point spectrum.

A torus with the two properties (i)+(ii) is nothing more and nothing less than a *quasi-periodic solution.*

If the quasi-periodic solution has property (iii), then questions related to linear stability and Lyapunov exponents "reduces" to a study of a linear system of constant coefficients, which permits (at least for finite-dimensional systems) to answer such questions. It also permits (at least for finite-dimensional systems) to construct higher order normal forms near the torus.

Reducibility is automatic in two cases: if the torus is one-dimensional (and phase-space is finite-dimensional) it is just a periodic solution, and (iii) is a general fact called Floquet theory; if the torus is Lagrangian (i.e. there is not ζ-part), then (iii) follows from (i)+(ii)[dlL01]. In general, however, it is a delicate property which is far from being completely understood.

KAM is a perturbation theory of KAM-tori. Not only is reducibility an important outcome but also an essential ingredient in the proof. It simplifies the iteration since it reduces all approximate linear equations to constant coefficients. But it does not come for free. It requires a lower bound on small divisors of the form

$$(**) \qquad |<k,\omega> +\Omega_a(\omega) \pm \Omega_b(\omega)|\,, \quad k \in \mathbb{Z}^{\mathcal{A}},\ a,b \in \mathcal{L},$$

where $\Omega_a(\omega),\ a \in \mathcal{L}$ are the imaginary parts of the eigenvalues of $JA(_o)$ The basic frequencies ω will be fixed during the iteration – that's what parameters are there for – but the normal frequencies will vary. Indeed the $\Omega_a(\omega)$ are perturbations of $|a|^2 + \hat{V}(a)$ which are not known a priori but are determined by the approximation process. [1]

The difficulty associated with the small divisors $(**)$ may be very large. There is a perturbation theory, often referred to as the Craig-Wayne scheme, which avoids this difficulty, but to a high cost: the approximate linear equations are no longer of constant coefficients. Moreover it gives persistence of the invariant tori but no reducibility.

[1] A lower bound on $(**)$ is strictly speaking not necessary at all for reducibility. It is necessary, however, in order to have reducibility with a reducing transformation close to the identity.

2.2. **Consequences of Theorem A.** The consequences of the theorem is that $\Phi(\{0,0\}\times\mathbb{T}^{\mathcal{A}})$ is a KAM-torus for $h_{\omega'}+\varepsilon f$. In order to see this it suffices to show that $\{\zeta=r=0\}$ is a KAM-torus for $k+\varepsilon g$,

$$k=<\omega,r>+\frac{1}{2}<\zeta,A(\omega)\zeta>\ .$$

Since

$$\frac{\partial g}{\partial\zeta}=\frac{\partial g}{\partial\varphi}=\frac{\partial g}{\partial r}=0$$

for $\zeta=r=0$, it follows that $\{\zeta=r=0\}$ is invariant with a flow $\varphi\mapsto\varphi+t\omega$. The linearized equations on this torus become

$$\begin{cases}\frac{d\hat\zeta}{dt}=JA(\omega)\hat\zeta+\varepsilon Ja(\varphi+t\omega,\omega)\hat r\\ \frac{d\hat r}{dt}=0\\ \frac{d\hat\varphi}{dt}=\varepsilon<{}^t a(\varphi+t\omega,\omega),\hat\zeta>+\varepsilon b(\varphi+t\omega,\omega)\hat r\end{cases}$$ [2]

where $a(\varphi,\omega)=\frac{\partial^2}{\partial r\partial\zeta}g(0,0,\varphi,\omega)$ and $b(\varphi,\omega)=\frac{\partial^2}{\partial r^2}g(0,0,\varphi,\omega)$.

These equations can be conjugated to constant coefficients if the imaginary part of the the eigenvalues of $JA(\omega)$,

$$\pm i\Omega_a(\omega),\quad a\in\mathcal{L},$$

are non-resonant with respect to ω. In order to see this we consider the equations

(i)
$$<\partial_\varphi Z_1(\varphi),\omega>=JAZ_1(\varphi)+\varepsilon Ja(\varphi),$$
which has a unique smooth solution if ω is Diophantine and
$$<k,\omega>\pm\Omega_a(\omega)\neq 0\qquad\forall k\in\mathbb{Z}^{\mathcal{A}},\ a\in\mathcal{L};$$

(ii)
$$<\partial_\varphi Z_2(\varphi),\omega>=-Z_2(\varphi)JA+\varepsilon\overset{*}{a}(\varphi)$$
which has a unique smooth solution under the same condition on ω;

(iii)
$$<\partial_\varphi Z_3(\varphi),\omega>=\varepsilon\ {}^t a(\varphi)Z_1(\varphi)+\varepsilon b(\varphi)-\varepsilon\beta$$
which has a smooth solution if ω is Diophantine and if we chose β such that the meanvalue of the right hand side is $=0$.

[2] t is used both as the independent time-variable and to denote transposition, without confusion we hope.

If we now take

$$Z(\varphi) = \begin{pmatrix} I & Z_1(\varphi) & 0 \\ 0 & I & 0 \\ Z_2(\varphi) & Z_3(\varphi) & I \end{pmatrix},$$

then $<Z(\varphi), \omega>=$

$$\begin{pmatrix} JA & \varepsilon Ja(\varphi) & 0 \\ 0 & 0 & 0 \\ \varepsilon^t|(|\varphi) & \varepsilon b(\varphi) & 0 \end{pmatrix} Z(\varphi) - Z(\varphi) \begin{pmatrix} JA & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varepsilon\beta & 0 \end{pmatrix},$$

so Z conjugates the linearized equations to

$$\begin{cases} \frac{d\hat{\zeta}}{dt} = JA(\omega)\hat{\zeta} \\ \frac{d\hat{r}}{dt} = 0 \\ \frac{d\hat{\varphi}}{dt} = \varepsilon\beta\hat{r} \end{cases}$$

which is constant coefficients.

The conditions on ω will hold if we restrict the set U' arbitrarily little.

If

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}, \tag{6}$$

then

$$C^{-1}JA(\omega)C = i \begin{pmatrix} {}^t\Omega(\omega) & 0 \\ 0 & -\Omega(\omega) \end{pmatrix},$$

since $\Omega(\omega) = \Omega_1(\omega) + i\Omega_2(\omega)$ is Hermitian. Moreover, there is a unitary matrix $D = D(\omega)$ such that

$${}^t\bar{D}\Omega(\omega)D = \mathrm{diag}(\Omega_a(\omega))$$

is a real diagonal matrix, and therefore

$$\begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}^{-1} i \begin{pmatrix} {}^t\Omega(\omega) & 0 \\ 0 & -\Omega(\omega) \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix}$$

$$= i \begin{pmatrix} \mathrm{diag}(\Omega_a(\omega)) & 0 \\ 0 & -\mathrm{diag}(\Omega_a(\omega)) \end{pmatrix}.$$

So the linearized equations on the torus have only quasi-periodic solutions and, hence, the torus is linearly stable.

2.3. **References.** For finite dimensional Hamiltonian systems the first proof of persistence of stable (i.e. vanishing of all Lyapunov exponents) lower dimensional invariant tori was obtained in [Eli85, Eli88] and there are now many works on this subjects. There are also many works on reducibility (see for example [Kri99, Eli01]) and the situation in finite dimension is now pretty well understood. Not so, however, in infinite dimension.

If $d = 1$ and the space-variable x belongs to a finite segment supplemented by Dirichlet or Neumann boundary conditions, this result was obtained in [Kuk88] (also see [Kuk93, Pös96]). The case of periodic boundary conditions was treated in [Bou96], using another multi–scale scheme, suggested by Fröhlich–Spencer in their work on the Anderson localization [FS83]. This approach, often referred to as the Craig-Wayne scheme, is different from KAM. It avoids the, sometimes, cumbersome condition $(**)$ but to a high cost: the approximate linear equations are not of constant coefficients. Moreover, it gives persistence of the invariant tori but no reducibility and no information on the linear stability. A KAM-theorem for periodic boundary conditions has recently been proved in [GY05] (with a perturbation F independent of x) and the perturbation theory for quasi-periodic solutions of one-dimensional Hamiltonian PDE is now sufficiently well developed (see for example [Kuk93, Cra00, Kuk00]).

The study of the corresponding problems for $d \geq 2$ is at its early stage. Developing further the scheme, suggested by Fröhlich–Spencer, Bourgain proved persistence for the case $d = 2$ [Bou98]. More recently, the new techniques developped by him and collaborators in their work on the linear problem has allowed him to prove persistence in any dimension d [Bou04]. (In this work he also treats the non-linear wave equation.)

3. The homological equation

3.1. **Normal form Hamiltonians.** This is a real Hamiltonian of the form

$$k = c(\omega) + <\omega, r> + \frac{1}{2}<\zeta, A(\omega)\zeta>,$$

where

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 \\ {}^t\Omega_2 & \Omega_1 \end{pmatrix}$$

is block-diagonal matrix with finite-dimensional blocks (we shall say more about these blocks in Section 4) and $\Omega(\omega) = \Omega_1(\omega) + i\Omega_2(\omega)$ is

Hermitian. Since $\Omega(\omega)$ is Hermitian the eigenvalues of $JA(\omega)$ are

$$\pm i\Omega_a(\omega) \quad a \in \mathcal{L},$$

where the $\Omega_a(\omega)$ are the (necessarily real) eigenvalues of $\Omega(\omega)$. (See the discussion in Section 2.2.)

We also suppose $A(\omega)$ to be close to

$$\begin{pmatrix} \mathrm{diag}(|a|^2 + \hat{V}(a) & \Omega_2 \\ 0 & \mathrm{diag}(|a|^2 + \hat{V}(a) \end{pmatrix}$$

and

$$\|\partial_\omega A(\omega)\| \leq \frac{1}{4}.$$

This implies that

$$\Omega_a(\omega) \approx |a|^2 + \hat{V}(a)$$

and $\mathcal{C}^1$-small in ω.

3.2. **The KAM-iteration.** Given a normal form Hamiltonian

$$h = <\omega, r> + \frac{1}{2} <\zeta, A(\omega)\zeta>$$

and a perturbation f. Let Tf be the Taylor polynomial

$$f(0,0,\varphi) + <\frac{\partial f}{\partial r}(0,0,\varphi), r> + <\frac{\partial f}{\partial \zeta}(0,0,\varphi), \zeta> + \frac{1}{2} <\zeta, \frac{\partial^2 f}{\partial \zeta^2}(0,0,\varphi)\zeta>$$

of f – it may also depend on ω.

If Tf was $= 0$ then $\{\zeta = r = 0\}$ would be a KAM-torus for $h + f$. But in general we only have

$$Tf \in \mathcal{O}(\varepsilon).$$

Suppose now there exist a Taylor polynomial s, i.e. $s = Ts$, and a normal form Hamiltonian

$$k = <\chi(\omega), r> + \frac{1}{2} <\zeta, B(\omega)\zeta> \quad \text{(moduo a constant)}$$

verifying

$$\{h, s\} = -Tf + k \tag{7}$$

– this equation is known as the *homological equation.*

Let Φ^t is the flow of

$$\begin{cases} \dot{\zeta} = J\frac{\partial s}{\partial \zeta}(\zeta, \varphi, r) \\ \dot{r} = -\frac{\partial s}{\partial \varphi}(\zeta, \varphi, r) \\ \dot{\varphi} = \frac{\partial s}{\partial r}(\zeta, \varphi, r). \end{cases}$$

If $s, k \in \mathcal{O}(\varepsilon)$, then $(\Phi^t - \mathrm{id}) \in \mathcal{O}(\varepsilon)$ and

$$\begin{aligned}(h+f)\circ\Phi^1 &= h + \varepsilon k + \textstyle\int_0^1 \frac{d}{dt}(h + tf + (1-t)k)\circ\Phi^t dt\\ &= h + k + \textstyle\int_0^1 (\{h + tf + (1-t)k, \varepsilon s\} + f - k)\circ\Phi^t dt\\ &= h + k + \textstyle\int_0^1 (\{tf + (1-t)k, s\} + f - Tf)\circ\Phi^t dt\\ &= h + k + [(f - Tf) + f_1].\end{aligned}$$

So Φ^1 transforms $h_\omega + f$ to a new normal form $h'_\omega = h_\omega + k$ plus a new perturbation f'. Since

$$T(f') \in \mathcal{O}(\varepsilon^2),$$

also

$$f' \in \mathcal{O}(\varepsilon^2)$$

when the domain is sufficiently restricted.

If we can solve the homological equation (7), not only for the normal form Hamiltonian h but also for all normal form Hamiltonians h', close to h, then we will be able to make an iteration which will converge to a solution as in Theorem A if the estimates a good enough. So the basic thing in KAM is to solve and estimate the solution of the homological equation.

It is clear from the discussion above that it is enough to solve a slightly weaker version of the homological equation, namely

$$\{h, s\} = -Tf + k + \mathcal{O}(\varepsilon^2). \tag{8}$$

3.3. **The homological equation.** We write s as

$$S_{01}(\varphi) + <S_{02}(\varphi), r> + <S_1(\varphi), \zeta> + \frac{1}{2}<\zeta, S_2(\varphi)\zeta>$$

and k as

$$c + <\chi, r> + \frac{1}{2}<\zeta, B\zeta> .$$

The homological equation (8) now decomposes into four linear equations.

$$\left\{\begin{array}{l} <\partial_\varphi S_{01}(\varphi), \omega> = -f(0,0,\varphi) + c + \mathcal{O}(\varepsilon^2);\\ <\partial_\varphi S_{02}(\varphi), \omega> = -\frac{\partial f}{\partial r}(0,0,\varphi) + \chi + \mathcal{O}(\varepsilon^2);\end{array}\right. \tag{9}$$

In these equations, we are forced to take

$$c = <f(0,0,\cdot)> \quad \text{and} \quad \chi = <\frac{\partial f}{\partial f}(0,0,\cdot)>,$$

where $<g>$ is the mean value

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(\varphi)d\varphi.$$

$$<\partial_\varphi S_1(\varphi),\omega> + JAS_1(\varphi) = -\frac{\partial f}{\partial \zeta}(0,0,\varphi) + \mathcal{O}(\varepsilon^2). \tag{10}$$

$$\begin{aligned} &<\partial_\varphi S_2(\varphi),\omega> + AJS_2(\varphi) - S_2(\varphi)JA \\ &\qquad = -\tfrac{\partial^2 f}{\partial \zeta^2}(0,0,\varphi) + B + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{11}$$

The most delicate of these equations is the last one which is related to reducibility. This is an equation for $gl(2,\mathbb{R})$-valued matrices

$$A \quad \text{and} \quad B = \begin{pmatrix} \Omega_1' & \Omega_2' \\ {}^t\Omega_2' & \Omega_1' \end{pmatrix}, \qquad \Omega' = \Omega_1' + i\Omega_2',$$

and

$$S_2(\varphi) \quad \text{and} \quad F(\varphi) = \frac{\partial^2 f}{\partial \zeta^2}(0,0,\varphi).$$

If we write $\tilde{F}(\varphi) = {}^tCF(\varphi)C$ and $\tilde{S}_2(\varphi) = {}^tCS_2(\varphi)C$, then equation (11) becomes

$$<\partial_\varphi \tilde{S}_2(\varphi),\omega> -i\begin{pmatrix} 0 & \Omega \\ {}^t\Omega & 0 \end{pmatrix} J\tilde{S}_2(\varphi) + i\tilde{S}_2(\varphi)J\begin{pmatrix} 0 & \Omega \\ {}^t\Omega & 0 \end{pmatrix} =$$

$$-\tilde{F}(\varphi) + i\begin{pmatrix} 0 & \Omega' \\ {}^t\Omega' & 0 \end{pmatrix} + \mathcal{O}(|f|^2).$$

This equation decouples into four equations for scalar-valued matrices. These are of the form

$$<\partial_\varphi R(\varphi),\omega> \pm i(\Omega R(\varphi) + R(\varphi){}^t\Omega) = G(\varphi) + \mathcal{O}(\varepsilon^2), \tag{12}$$

for the diagonal terms, and of the form

$$<\partial_\varphi R(\varphi),\omega> \pm i(\Omega R(\varphi) - R(\varphi)\Omega) = G(\varphi) - \Omega' + \mathcal{O}(\varepsilon^2) \tag{13}$$

for the off-diagonal terms.

The last equation is underdetermined and there are several possible choices of Ω'. One such choice would be $<G>$ which would give an Hermitian matrix, but in general not a block diagonal matrix. So the Hamiltonian $h' = h + k$ would not be on normal form. Instead we shall make the "smaller" choice.

Due to the exponential decay of the second order derivatives of the Hamiltonian (discussed in Section 1.3) the matrix G verifies

$$|G(\varphi)_a^b| \lesssim \varepsilon e^{-\gamma|a-b|} \quad a,b \in \mathcal{L},$$

and we can truncate the matrices away from the diagonal at distance

$$\Delta' \approx \log(\frac{1}{\varepsilon}).$$

We then take

$$(\Omega')_a^b = \begin{cases} <G_a^b> & |a| = |b|,\ |a-b| \leq \Delta' \\ 0 & |a| \neq |b| \end{cases} \tag{14}$$

Since the left hand side of the equations (9-13) are linear operators with constant coefficients, equations (9-14) can be solved in Fourier series and to get a solution we must prove the convergence of these Fourier series and estimate the solution. This requires good estimates on the small divisors, i.e. the eigenvalues of the linear operators in the left hand side.

3.4. **Small Divisors and the second Melnikov condition.** Since the equations are to be solved only modulo $\mathcal{O}(\varepsilon^2)$ and since all functions are analytic in φ, we can truncate all Fourier series to order

$$\Delta' \approx \log(\frac{1}{\varepsilon}).$$

We want to bound the eigenvalues (in absolute value) in the left hand side from below by some quantity κ which should be small but much larger than ε, say

$$\kappa = \varepsilon^{\exp}$$

for some small exponent exp.

For equation (9), the eigenvalues of the left hand side operator are

$$i <k,\omega> \quad k \in \mathbb{Z}^{\mathcal{A}},\ 0 < |k| \leq_D' .$$

These are all larger (in absolute value) than κ for all $\omega \in U$ except on a small set of Lebesgue measure

$$\lesssim (\Delta')^{\#\mathcal{A}}\kappa.$$

The eigenvalues in equation (10) are

$$i <k,\omega> +i\Omega_a(\omega) \quad k \in \mathbb{Z}^{\mathcal{A}},\ |k| \leq_D', \quad a \in \mathcal{L},$$

where the $\Omega_a(\omega)$:s are the eigenvalues of $A(\omega)$. By the assumption on $A(\omega)$,

$$\Omega_a(\omega) \approx |a|^2 + \hat{V}(a)$$

and is $\mathcal{C}^1$-small in ω. Therefore there are only finitely many eigenvalues which are not large, and these can be controlled by an appropriate choice of ω.

Equation (12) is treated in the same way.

It is the equation (13) which give rise to serious problems. If we define Ω' by (14) and take into account the exponential decay of the matrices, then he eigenvalues of equation (13) are

$$i(\Omega_a(\omega) - \Omega_b(\omega))) \quad |a-b| \leq \Delta',\ |a| \neq |b|,$$

(which are all $\gtrsim 1$ by assumption (5)) and

$$\begin{cases} i <k,\omega> +i(\Omega_a(\omega)-\Omega_b(\omega))) \\ k\in\mathbb{Z}^{\mathcal{A}},\ 0<|k|\le\Delta', |a-b|\le\Delta'. \end{cases} \tag{15}$$

In one space dimension $d=1$ we have

$$|\Omega_a(\omega)-\Omega_b(\omega)|\to\infty$$

when $|a|\to\infty$, $|a-b|\le\Delta'$, except for $a=b$. Therefore there are only finitely many eigenvalues which are not large, and these can be controlled by an appropriate choice of ω.

But in dimension $d\ge 2$ there are infinitey many eigenvalues which are not large. How to control (15) – known as the second Melnikov condition – is the main difficulty in the proof. But before we turn to this question we shall discuss more closely the normal form.

4. Blocks and Lipschitz-domains

4.1. **Blocks.** In this section $d\ge 2$. For a non-negative integer Δ we define an *equivalence relation* on $\mathcal{L}$ generated by the pre-equivalence relation

$$a\sim b \iff \begin{cases} |a|^2=|b|^2 \\ |a-b|\le\Delta. \end{cases}$$

Let $[a]_\Delta$ denote the equivalence class (*block*) of a, and let $\mathcal{E}_\Delta$ be the set of equivalence classes. It is trivial that each block $[a]$ is finite with cardinality

$$\lesssim |a|^{d-1}$$

that depends on a. But there is also a uniform Δ-dependent bound.

Lemma 4.1. *Let*

$$d_\Delta=\sup_a(\mathrm{diam}[a]_\Delta).$$

Then

$$d_\Delta\lesssim\ \Delta^{\frac{(d+1)!}{2}}.$$

Proof. We give the proof in dimension $d=2$, the general case being treated in Section 4 of [EK06].

It suffices to consider the case when there are $a,b,c\in[a]_\Delta$ such that $a-b$ and $a-c$ are linearly independent and

$$|a-b|,|a-c|_le\Delta.$$

(If not, then $[a]_\Delta=\{a,b\}$ and the result is obvious.)

Since $|a|^2=|b|^2=|c|^2$ it follows that

$$\begin{cases} <a,a-b>=\frac{1}{2}|a-b|^2 \\ <a,a-c>=\frac{1}{2}|a-C|^2 \end{cases}$$

Since $a - b$ and $a - c$ are integervalued independent vectors it follow from this equation that

$$|a| \lesssim \Delta^3.$$

□

The blocks $[a]_\Delta$ have a rigid structure when $|a|$ is large. For a vector $c \in \mathbb{Z}^d \setminus 0$ let

$$a_c \in (a + \mathbb{R}c) \cap \mathbb{Z}^d$$

be the lattice point b on the line $a + \mathbb{R}c$ with smallest norm – if there are two such b's we choose the one with $<b, c> \geq 0$.

Lemma 4.2. *Given a and $c \neq 0$ in $\mathbb{Z}^d$. For all t, such that*

$$|a + tc| \geq d_\Delta^2(|a_c| + |c|)\,|c|\,,$$

the set $[a + tc]_\Delta - (a + tc)$ is independent of t and $\perp$ to c.

Proof. It suffices to prove this for $a = a_c$.

Let $b \in [a + t]\_D$ for some fixed t as in the lemma. This implies, be Lemma **??**, that $|b| \leq d_\Delta$ and that $|b + a + tc|^2 = |a + tc|^2$. This last relation is equivalent to

$$2t <b, c> +2 <b, a> +|b|^2 = 0.$$

If $<b, c> \neq 0$, then

$$\begin{aligned} |a + tc| &\leq |a| + |t <b, c> ||c|^2 \\ &= |a| + | <b, a> +\tfrac{1}{2}|b|||c| \\ &\leq (1 + d_\Delta)|a||c| + \tfrac{1}{2}d_\Delta^2|c|^2, \end{aligned}$$

but this is impossible under the assumption on $a + tc$.

Therefore $<b, c>= 0$ and hence $[a + tc]_\Delta - (a + tc) \perp c$. Moreover $|b + a + sc|^2 = |a + sc|^2$ for all s, so if $|b| \leq \Delta$, then

$$[b + a + sc)_\Delta = [a + sc]_\Delta \quad \forall s.$$

To conclude, let $b_0 = a, b_1, \ldots, b_n$ be the elements of $[a]_\Delta$ ordered in such a way that $b_{j+1} - b_j|_l e\Delta$ for all j. Then the preceding argument shows that

$$[b + a + sc)_\Delta = [a + sc]_\Delta \quad \forall s,_\forall j.$$

□

Description of blocks when $d = 2, 3$. For $d = 2$, we have outside $\{|a| :\leq d_\Delta \approx \Delta^3\}$

- ⋆ $\operatorname{rank}[a]_\Delta = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \leq \Delta$ – then $[a]_\Delta = \{a, a - b\}$;
- ⋆ $\operatorname{rank}[a]_\Delta = 0$ otherwise – then $[a]_\Delta = \{a\}$.

For $d = 3$, we have outside $\{|a| :\leq d_\Delta \approx \Delta^{12}\}$

- ⋆ $\mathrm{rank}[a]_\Delta = 2$ if, and only if, $a \in \frac{b}{2} + b^\perp \cap \frac{c}{2} + c^\perp$ for some $0 < |b|, |c| \le 2\Delta$ linearly independent – then $[a]_\Delta \supset \{a, a-b, a-c\}$;
- ⋆ $\mathrm{rank}[a]_\Delta = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \le \Delta$ – then $[a]_\Delta = \{a, a-b\}$;
- ⋆ $\mathrm{rank}[a]_\Delta = 0$ otherwise – then $[a]_\Delta = \{a\}$.

4.2. **Normal form matrices and Hamiltonians.** We say that a (scalar-valued) matrix $X : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is on normal form – denoted $\mathcal{NF}_\Delta$ – if

(i) X is Hermitian;
(ii) X is *block-diagonal* over over $\mathcal{E}_\Delta$, i.e.

$$X_a^b = 0 \quad \text{if} \quad [a]_\Delta \ne [b]_\Delta.$$

We say that our normal form Hamiltonians

$$h = c + \langle \omega, r \rangle + \frac{1}{2} \langle \zeta, A(\omega)\zeta \rangle,$$

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 \\ {}^t\Omega_2 & \Omega_1 \end{pmatrix},$$

is $\mathcal{NF}_\Delta$ if $\Omega = \Omega_1 + i\Omega_2$ is $\mathcal{NF}_\Delta$.

Clearly if h is $\mathcal{NF}_\Delta$ for some $\Delta \le \Delta'$, then by the choice of Ω' in (14) $h' = h + k$ is $\mathcal{NF}_{\Delta'}$, where

$$k = c + \langle \chi, r \rangle + \frac{1}{2} \langle \zeta, B\zeta \rangle$$

is determined in Section 3.3.

4.3. **Lipschitz domains.** For a non-negative constant Λ and for any $c \in \mathbb{Z}^d \setminus 0$, let the Lipschitz domain

$$D_\Lambda(c) \subset \mathcal{L} \times \mathcal{L}$$

be the set of all (a, b) such that there exist $a', b' \in \mathbb{Z}^d$ and $t \ge 0$ such that

$$\begin{cases} |a = a' + tc| & \ge \Lambda(|a'| + |c|)\,|c| \\ |b = b' + tc| & \ge \Lambda(|b'| + |c|)\,|c| \end{cases}$$

and

$$\frac{|a|}{|c|}, \quad \frac{|b|}{|c|} \ \ge\ 2\Lambda^2.$$

The Lipschitz domains are not so easy to grasp, but it is easy to verify

Lemma 4.3. *Let* $\Lambda \ge 3$.

(i) *If* $a = a' + tc| \leq \Lambda(|a'| + |c|)|c|,\ t \geq 0,$, *then*

$$\frac{|a|}{|c|} \approx \frac{<a,c>}{|c|^2} \approx t \gtrsim \Lambda|c|.$$

(ii) *If* $a = a' + t_0c| \leq \Lambda(|a'| + |c|)|c|, \tau_0 \geq 0$, *then*

$$|a' + tc|^2 \geq |a' + t_0c|^2 + (t - t_0)^2|c|^2 \quad \forall t \geq t_0.$$

In particular, if $(a, b) \in D_\Lambda(c)$, *then*

$$(a + tc, b + tc) \in D_\Lambda(c) \quad \forall t \geq 0.$$

Proof. (i) The inequality $|a' + tc| \leq |a'| + t|c| \leq (|a'| + t)|c|$ gives immediately that $t \geq \Lambda|c|$.

It also gives

$$\Lambda(|a'| + |c|) \leq |a'| + t,$$

which implies that

$$|a'| \leq \frac{t}{\Lambda - 1}.$$

Since

$$|\frac{|a|}{|c|} - t|, |\frac{<a,c>}{|c|^2} - t| \leq \frac{|a'|}{|c|}$$

we are done.

(ii) Let $s = t - t_0$. Then

$$|a + sc|^2 = |a|^2 + s^2|c|^2 + 2s <a,c>$$

and

$$2s <a,c> = 2st_0|c|^2 + <a',c> \geq 2st_0(|c|^2 - \frac{|a'||c|}{t_0})$$

which is ≥ 0. □

A bit more complicated is

Lemma 4.4. *For any* $|a| \gtrsim \Lambda^{2d-1}$, *there exist* $c \in \mathbb{Z}^d$,

$$0 < |c| \lesssim \Lambda^{d-1},$$

such that

$$|a| \geq \Lambda(|a_c| + |c|)\,|c|\,,\ <a,c> \geq 0.$$

Proof. For all $K \gtrsim 1$ there is a $c \in \mathbb{Z}^d \cap \{|x| \leq K\}$ such that

$$\delta = dist(c, \mathbb{R}a) \leq C_1(\frac{1}{K})^{\frac{1}{d-1}}$$

where C_1 only depends on d.

To see this we consider the segment $\Gamma = [0, \frac{K}{|a|}a]$ in $\mathbb{R}^d$ and a tubular neighborhood Γ_ε of radius ε:

$$\mathrm{vol}(\Gamma_\varepsilon) \approx K\varepsilon^{d-1}.$$

The projection of $\mathbb{R}^d$ onto $\mathbb{T}^d$ is locally injective and locally volume-preserving. If $\varepsilon \gtrsim (\frac{1}{K})^{\frac{1}{d-1}}$, then the projection of Γ_ε cannot be injective (for volume reasons), so there are two different points $x, x' \in \Gamma_\varepsilon$ such that

$$x - x' = c \in Z^d \setminus 0.$$

Then

$$|a_c| \lesssim \frac{|a|}{|c|}\delta.$$

Now

$$\Lambda(|a_c| + |c|)\,|c| \le 2\Lambda K^2 + C_2 \frac{\Lambda}{K^{\frac{1}{d-1}}}\,|a|\,.$$

If we choose $K = (2C_2\Lambda)^{d-1}$, then this is $\le |a|$. □

The most important property is that finitely many Lipschitz domains cover a "neighborhood of ∞" in the following sense.

Corollary 4.5. *For any $\Lambda, N > 1$, the subset*

$$\{|a| + |b| \gtrsim \Lambda^{2d-1}\} \cap \{|a - b| \le N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

is contained in

$$\bigcup_{0<|c|\lesssim\Lambda^{d-1}} D_\Omega(c)$$

for any

$$\Omega \le \frac{\Lambda}{N+1} - 1.$$

Proof. Let $|a| \gtrsim \Lambda^{2d-1}$. Then there exists $0 < |c| \lesssim \Lambda^{d-1}$ such that $|a| \ge \Lambda(|a_c| + |c|)\,|c|$. Clearly (because $d \ge 2$)

$$\frac{|a|}{|c|} \ge 2\Lambda^2 \ge 2\Omega^2.$$

If we write $a = a_c + tc$ then $b = a_c + b - a + tc$. Then

$$\begin{aligned} \Omega(|a_c + b - a| + |c|)|c| &\le \Omega(|a_c| + |c|)|c| + \Omega(|b-a||c| \\ &\le \Lambda(|a_c| + |c|)|c| - |b-a||c| \\ &\le |a| - |b-a| \le |b|, \end{aligned}$$

if and only if

$$(\Omega - \Lambda)(|a_c| + |c|) \ge (\Omega + 1)|b - a|,$$

which holds by the assumption on Ω. Moreover

$$\frac{|b|}{|c|} \ge \frac{|a|}{|c|} - N \ge 2\Lambda^2 - N \ge 2\Omega^2.$$

□

5. Töplitz-Lipschitz matrices (d = 2)

5.1. **Töplitz at ∞.** We say that a matrix

$$X : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$$

has a Töplitz-limit at ∞ in the direction c if, for all a, b

$$\lim_{t\to\infty} X_{a+tc}^{b+tc} \ \exists \ = X_a^b(c).$$

$X(c)$ is a new matrix which is Töplitz in the direction c, i.e.

$$X_{a+c}^{b+c}(c) = X_a^b(c).$$

We say that X is Töplitz at ∞ if it has a Töplitz-limit in any direction c.

Example. Consider the equation (13) for the unperturbed Hamiltonian, i.e.

$$\Omega = \mathrm{diag}(|a|^2 + \hat{V}(a)).$$

Then

$$\hat{R}(k)_a^b = \frac{\hat{G}(k)_a^b}{i(<k,\omega> + |a|^2 - |b|^2 + \hat{V}(a) - \hat{V}(b))}$$

and if the small divisors are all $\neq 0$ then $\hat{R}(k)$ is a well-defined matrix $\mathcal{L} \times \mathcal{L} \to \mathbb{C}$. Replacing a, b by $a + tc, b + tc$ and letting $t \to \infty$ we see two different cases. If $<a - b, c>\neq 0$ then the limit exist and is $= 0$ as long as $|\hat{G}(k)_{a+tc}^{b+tc}|$ is bounded. If $<a - b, c>= 0$ then the limit exist as long as $|\hat{G}(k)_{a+tc}^{b+tc}|$ has a limit:

$$\hat{R}(k)_a^b(c) = \frac{\hat{G}(k)_a^b(c)}{i(<k,\omega> + |a|^2 - |b|^2)}.$$

Hence the matrix $\hat{R}(k)$ is Töplitz at ∞ if $\hat{G}(k)$ is Töplitz at ∞.

If $X : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is a Töplitz matrix, let us consider the block decomposition of X into finite-dimensional submatrices

$$X_{[a]_\Delta}^{[b]_\Delta} = \{X_a^b : a \in [a]_\Delta, b_i n [b]_\Delta\}.$$

The dimension of $X_{[a]_\Delta}^{[b]_\Delta}$ varies with a and b, but if $(a, b) \in D_\Lambda(c)$, $\Lambda \geq d_\Delta^2$, then (by Lemma 4.2)

$$X_{[a]_\Delta}^{[b]_\Delta}(tc) =: X_{[a+tc]_\Delta}^{[b+tc]_\Delta}$$

is a well-defined matrix which depends on the parameter $t \geq$ and has a limit as $t \to \infty$.

5.2. **Töplitz-Lipschitz matrices.** We define the supremum-norm

$$|X|_\gamma = \sup_{a,b\in\mathcal{L}} |X|_a^b e^{\gamma|a-b|},$$

and if X is Töplitz at ∞, the Lipschitz-constant

$$\mathrm{Lip}_{\Lambda,\gamma} X = \sup_{c\in\mathbb{Z}^d\setminus 0} \sup_{(a,b)\in D_\Lambda(c)} |X_a^b - X_a^b(c)| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

and the Lipschitz-norm

$$< X >_{\Lambda,\gamma} = \mathrm{Lip}_{\Lambda,\gamma} X + |X|_\gamma .$$

We say that the matrix X is *Töplitz-Lipschitz* if

$$< X >_{\Lambda,\gamma} < \infty$$

for some Λ, γ.

Example. Consider $\hat{R}(k)$ from the example above. If

$$(a,b) \in D_\Lambda(c), \quad \Lambda \geq 3,$$

then

$$|a = a' + tc| \leq \Lambda(|a'| + ||c|)|c| \quad \text{and} \quad |b = b' + tc| \leq \Lambda(|b'| + ||c|)|c|.$$

By Lemma **??** we have

$$\frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx t \geq \Lambda.$$

If $<a-b,c> \neq 0$ then

$$\left|\hat{R}(k)_a^b - 0\right| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

$$\approx \left| \frac{\hat{G}(k)_a^b}{<a-b,c> + \frac{1}{t}(<k,\omega> + |a'|^2 - |b'|^2 + \hat{V}(a) - \hat{V}(b))} \right| e^{\gamma|a-b|}$$

which is

$$\approx \left| \frac{\hat{G}(k)_a^b}{<a-b,c>} \right| e^{\gamma|a-b|} \lesssim |G|_\gamma$$

if Λ, hence t, is sufficiently large.

If $<a-b,c> = 0$ then

$$\left|\hat{R}(k)_a^b - \hat{R}(k)(c)_a^b\right| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

$$\lesssim \left| \frac{1}{<k,\omega> + |a'|^2 - |b'|^2} \right| \mathrm{Lip}_{\Lambda,\gamma}(\hat{G}(k)) + \left| \frac{1}{<k,\omega> + |a'|^2 - |b'|^2} \right|^2 \left|\hat{G}(k)\right|_\gamma$$

if Λ, hence t, is sufficiently large. Here we have used the decay of $\hat{V}$ to bound

$$|\hat{V}(a'+tc)-\hat{(}b'+tc)|t\lesssim 1.$$

In particular, the matrix $\hat{R}(k)$ is Töplitz-Lipschitz if $\hat{G}(k)$ is Töplitz-Lipschitz.

5.3. **A multiplicative formula.** In Section 2 of [EK06] we prove the following multiplicative property.

Proposition 5.1. *Let $X_1, X_2 : \mathcal{L}\times\mathcal{L}\to\mathbb{C}$ be Töplitz-Lipschitz matrices with exponential decay off-diagonal, i.e.*

$$|X_j|_\gamma<\infty \quad j=1,2,\ \gamma>0.$$

Then X_1X_2 is Töplitz-Lipschitz and

$$<X_1X_2>_{\Lambda+3,\gamma'}\lesssim \Lambda^2(\tfrac{1}{\gamma-\gamma'})^{d+1}[<X_1>_{\Lambda,\gamma_1}|X_2|_{\gamma_2}+|X_1|_{\gamma_1}<X_2>_{\Lambda,\gamma_2}],$$

where one of γ_1,γ_2 is $=\gamma$ and the other one is $=\gamma'$.

Let $X_1,\ldots,X_n : \mathcal{L}\times\mathcal{L}\to\mathbb{C}$ be Töplitz-Lipschitz matrices with exponential decay off-diagonal, i.e.

$$|X_j|_\gamma<\infty \quad j=1,\ldots,n,\ \gamma>0.$$

Then $X_1\cdots X_n$ is Töplitz-Lipschitz and

$$<X_1\cdots X_n>_{\Lambda+3,\gamma'}\lesssim \Lambda^2(\tfrac{1}{\gamma-\gamma'})^{d+1}[<X_1>_{\Lambda,\gamma_1}|X_2|_{\gamma_2}+|X_1|_{\gamma_1}<X_2>_{\Lambda,\gamma_2}],$$

where one of γ_1,γ_2 is $=\gamma$ and the other one is $=\gamma'$.

Notice that the second estimate is not an interation of the first estimate.

Linear differential equation. Consider the linear system

$$\begin{cases} \frac{d}{dt}X=A(t)X \\ X(0)=I. \end{cases}$$

where $A(t)$ is Töplitz-Lipschitz with exponential decay. The solution verifies

$$X(t_0)=I+\sum_{n=1}^{\infty}\int_0^{t_0}\int_0^{t_1}\ldots\int_0^{t_{n-1}}A(t_1)A(t_2)\ldots A(t_n)dt_n\ldots dt_2dt_1.$$

Using Proposition 5.1 we get for $\gamma'<\gamma$

$$<X(t)-I>_{\Lambda+6,\gamma'}\lesssim \Lambda^2(\tfrac{1}{\gamma-\gamma'})|t|\exp(\text{cte.}(\tfrac{1}{\gamma-\gamma'})^d|t|\alpha(t))\sup_{|s|\le|t|}<A(s)>_{\Lambda,\gamma},$$

where

$$\alpha(t) = \sup_{0 \le |s| \le |t|} |A(s)|_\gamma .$$

6. Estimates of small divisors

6.1. A basic estimate.

Lemma 6.1. *Let $f : I =]-1, 1[\to \mathbb{R}$ be of class $\mathcal{C}^n$ and*

$$\left| f^{(n)}(t) \right| \ge 1 \quad \forall t \in I.$$

Then, $\forall \varepsilon > 0$, the Lebesgue measure of $\{t \in I : |f(t)| < \varepsilon\}$ is

$$\le \text{cte}.\varepsilon^{\frac{1}{n}},$$

where the constant only depends on n.

Proof. We have $\left| f^{(n)}(t) \right| \ge \varepsilon^{\frac{0}{n}}$ for all $t \in I$. Since

$$f^{(n-1)}(t) - f^{(n-1)}(t_0) = \int_{t_0}^{t} f^{(n)}(s) ds,$$

we get that $\left| f^{(n-1)}(t) \right| \ge \varepsilon^{\frac{1}{n}}$ for all t outside an interval of length $\le 2\varepsilon^{\frac{1}{n}}$. By induction we get that $\left| f^{(n-j)}(t) \right| \ge \varepsilon^{\frac{j}{n}}$ for all t outside 2^{j-1} intervals of length $\le 2\varepsilon^{\frac{1}{n}}$. $j = n$ gives the result. □

Remark. The same is true if

$$\max_{0 \le j \le n} \left| f^{(j)}(t) \right| \ge 1 \quad \forall t \in I$$

and $f \in \mathcal{C}^{n+1}$. In this case the constant will depend on $|f|_{\mathcal{C}^{n+1}}$.

Let $A(t)$ be a real diagonal $N \times N$-matrix with diagonal components a_j which are $\mathcal{C}^1$ on $I =]-1, 1[$ and

$$a_j'(t) \ge 1 \qquad j = 1, \dots, N, \ \forall t \in I.$$

Let $B(t)$ be a Hermitian $N \times N$-matrix of class $\mathcal{C}^1$ on $I =]-1, 1[$ with

$$\|B'(t)\| \le \frac{1}{2} \quad \forall t \in I.$$

Lemma 6.2. *The Lebesgue measure of the set*

$$\{t \in I : \min_{\lambda(t) \in \sigma(A(t)+B(t))} |\lambda(t)| < \varepsilon\}$$

is

$$\le \text{cte}.N\varepsilon,$$

where the constant is independent of N.

Proof. Assume first that $A(t)+B(t)$ is analytic in t. Then each eigenvalue $\lambda(t)$ and its (normalized) eigenvector $v(t)$ are analytic in t, and

$$\lambda'(t) = <v(t), (A'(t)+B'(t))v(t)>$$

(scalar product in $\mathbb{C}^N$). Under the assumptions on A and B, this is $\geq 1-\frac{1}{2}$. Lemma 6.1 applied to each eigenvalue $\lambda(t)$ gives the result.

If B is non-analytic we get the same result by analytic approximation. □

We now turn to the main problem.

6.2. **The second Melnikov condition (**d = 2**).** Consider a matrix $X : \mathcal{L}\times\mathcal{L}\to\mathbb{C}$ that depends $\mathcal{C}^1$ on $\omega\in U$. If $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at $_i$ for all $\omega\in U$ then we define

$$<H>_{\{\substack{\Lambda\\U}\}} = \sup_U(<H(\omega>_\Lambda, <\partial_o H(\omega>_\Lambda).$$

If this norm is finite, then, clearly, the convergence to the Töplitz-limits is uniform in ω both for $(H(\omega)$ and $\partial_\omega H(\omega)$.

Proposition 6.3. *Let $\Delta'>1$ and $0<\kappa<1$. Assume that U verifies (3), that*

$$\Omega = \mathrm{diag}(|a|^2+\hat{V}(a) : a\in\mathcal{L})$$

verifies (4) and that $H : \mathcal{L}\times\mathcal{L}\to\mathbb{C}$ verifies

$$\|\partial_\omega H(\omega)\| \leq \frac{1}{4} \quad \omega\in U. \tag{16}$$

($\|\cdot\|$ is the operator norm.) Assume also that $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega\in U$.

Then there exists a subset $U'\subset U$,

$$Leb(U\setminus U') \leq$$
$$\text{cte.}\max(\Delta', d_\Delta^2, \Lambda)^{exp+\#\mathcal{A}-1}(C_1+<H>_{\{\substack{\Lambda\\U}\}})^d\kappa^{\frac{1}{3}}C_1^{\#\mathcal{A}-1},$$

such that, for all $\omega\in U'$, $0<|k|\leq\Delta'$ and all

$$\mathrm{dist}([\mathrm{a}]_\Delta, [\mathrm{b}]_\Delta) \leq \Delta' \tag{17}$$

we have

$$|<k,\omega> + \alpha(\omega)-\beta(\omega)| \geq \kappa \quad \forall \begin{cases} \alpha(\omega)\in\sigma((\Omega+H)(\omega)_{[a]_\Delta}) \\ \beta(\omega)\in\sigma((\Omega+H)(\omega)_{[b]_\Delta}). \end{cases} \tag{18}$$

Moreover the κ-neighborhood of $U\setminus U'$ satisfies the same estimate.

The exponent exp is a numerical constant. The constant cte. *depends on $\#\mathcal{A}$ and on C_2, C_3.*

Proof. The proof goes in the following way: first we prove an estimate in a large finite part of $\mathcal{L}$ (this requires parameter restriction); then we assume an estimate "at ∞" of $\mathcal{L}$ and we prove, using the Lipschitz-property, that this estimate propagate from "∞" down to the finite part (this requires no parameter restriction); in a third step we have to prove the assumption at ∞.

Let us notice that it is enough to prove the statement for $\Delta' \geq \max(\Lambda, d_\Delta^2)$. We let $[\ ]$ denote $[\ ]_\Delta$. Let $\Omega \approx (\Delta')^2$.

For each $k, [a]_\Delta, [b]_\Delta$ it follows by Lemma 6.2 the set of ω such that

$$|{<}k, \omega{>} + \alpha(\omega) - \beta(\omega)| < \kappa$$

has Lebesgue measure

$$\lesssim d_\Delta^d \frac{\kappa}{|\kappa|} C_1^{\#\mathcal{A}-1}.$$

1. Finite part. For the finite part, let us suppose a belongs to

$$\{a \in \mathcal{L} : |a| \lesssim (C_1 + \frac{1}{\kappa_1} d_\Delta^d < H >_{\{\substack{\Lambda \\ U}\}}) \Omega^{2d-1}\}, \tag{19}$$

[3] where $\kappa_1 = \kappa^{\frac{1}{3}} = \kappa^{\frac{1}{d+1}}$. These are finitely many possibilities and $(18)_\kappa$ is fulfilled, for all $[a]$ satisfying (19), all $[b]$ with $|a - b| \lesssim \Delta'$ and all $0 < |k| \leq \Delta'$, outside a set of Lebesgue measure

$$\lesssim (C_1 + d_\Delta^d < H >_{\{\substack{\Lambda \\ U}\}})^d \Omega^{d(2d-1)} (\Delta')^{d+\#\mathcal{A}-1} \frac{\kappa}{\kappa_1^d} C_1^{\#\mathcal{A}-1}. \tag{20}$$

Let us now get rid of the diagonal terms $\hat{V}(a, \omega) = \Omega_a(\omega) - |a|^2$ which, by (4), are

$$\leq C_2 e^{-|a| C_3}.$$

We include them into H. Since they are diagonal, H will remain on normal form. Due to the exponential decay of $\hat{V}$, H and $\partial_\omega H$ will remain Töplitz at ∞. The Lipschitz norm gets worse but this is innocent in view of the estimates. Also the estimate of $\partial_\omega H(\omega)$ gets worse, but if a is outside (19) then condition (16) remains true with a slightly worse bound, say

$$\|\partial_\omega H(\omega)\| \leq \frac{3}{8}, \quad \omega \in U.$$

So from now on, a is outside (19) and

$$\Omega_a = |a|^2.$$

[3]In this proof $\lesssim$ depends on $\#\mathcal{A}$ and on C_2, C_3.

2. Condition at ∞. For each vector $c \in \mathbb{Z}^d$ such that $0 < |c| \lesssim \Omega^{d-1}$, we suppose that the Töplitz limit $H(c,\omega)$ verifies $(18)_{\kappa_1}$ for (17) and for

$$([a]-[b]) \perp c. \tag{21}$$

It will become clear in the next part why we only need $(18)_{\kappa_1}$ and (17) under the supplementary restriction (21).

3. Propagation of the condition at ∞. We must now prove that for $|b-a| \lesssim \Delta'$ and an $a \in \mathcal{L}$ outside (19), $(18)_\kappa$ is fulfilled.

By the Corollary 4.5 we get

$$(a,b) \in \bigcup_{0<|c|\lesssim\Omega^{d-1}} D_{\Omega'}(c), \quad \Omega' \approx \frac{\Omega}{\Delta'}.$$

Fix now $0 < |c| \lesssim \Omega^{d-1}$ and $(a,b) \in D_{\Omega'}(c)$. By Lemma 4.2 – notice that $\Omega' \geq d_\Delta^2$ –

$$[a+tc] = [a] + tc \quad \text{and} \quad [b+tc] = [b] + tc$$

for $t \geq 0$ and

$$[a]-a,\ [b]-b \ \perp\ c.$$

It follows that

$$\lim_{t\to\infty} H(\omega)_{[a+tc]} = H(c,\omega)_{[a]} \quad \text{and} \quad \lim_{t\to\infty} H(\omega)_{[b+tc]} = H(c,\omega)_{[b]}.$$

The matrices $\Omega_{[a+tc]}$ and $\Omega_{[b+tc]}$ do not have limits as $t \to \infty$. However, for any $(\#[a] \times \#[b])$-matrix X,

$$\Omega_{[a+tc]}X - X\Omega_{[b+tc]} \ = \ \Omega_{[a]}X - X\Omega_{[b]} + 2t <a-b,c> X$$

for $t \geq 0$, and we must discuss two different cases according to if $< c, b-a >= 0$ or not.

Consider for $t \geq 0$ a pair of continuous eigenvalues

$$\begin{cases} \alpha_t \in \sigma((\Omega + H(\omega))_{[a+tc]}) \\ \beta_t \in \sigma((\Omega + H(\omega))_{[b+tc]}) \end{cases}$$

Case I: $<c, b-a>= 0$. Here

$$(\Omega + H(\omega))_{[a+tc]}X - X(\Omega + H(\omega))_{[b+tc]}$$

equals

$$(|a|^2 + H(\omega))_{[a+tc]}X - X(|b|^2 + H(\omega))_{[b+tc]}$$

– the linear and quadratic terms in t cancel!

By continuity of eigenvalues,

$$\lim_{t\to\infty}(\alpha_t - \beta_t) = (\alpha_\infty - \beta_\infty),$$

where

$$\begin{cases} \alpha_\infty \in \sigma((|a|^2 + H(c,\omega))_{[a]}) \\ \beta_\infty \in \sigma((|b|^2 + H(c,\omega))_{[b]}) \end{cases}$$

Since $[a]$ and $[b]$ verify (21), our assumption on $H(c,\omega)$ implies that $(\alpha_\infty - \beta_\infty)$ verifies $(18)_{\kappa_1}$.

For any two $a, a' \in [a]$ we have, since a violates (19) and $|a-a'| \le d_\Delta$,

$$\frac{|a'|}{|c|} \approx \frac{|a|}{|c|}.$$

Hence

$$\left\|H(\omega)_{[a]} - H(c,\omega)_{[a]}\right\| \frac{|a|}{|c|} \lesssim d_\Delta^d < H >_{\{\Lambda \atop U\}},$$

because $\Delta' \ge \Lambda$, and the same for $[b]$. Recalling that a and, hence, b violate (19) this implies

$$\left\|H(\omega)_{[d]} - H(c,\omega)_{[d]}\right\| \le \frac{\kappa_1}{4}, \quad d = a, b.$$

By Lipschitz-dependence of eigenvalues (of Hermitian operators) on parameters, this implies that

$$|(\alpha_0 - \beta_0) - (\alpha_\infty - \beta_\infty)| \le \frac{\kappa_1}{2}$$

and we are done.

Case II: $<c, b-a> \ne 0$. We write $a = a_c + \tau c$, where is the lattice point on the line $a + \mathbb{R}c$ with smallest norm – if there are two such points we choose the one with $<a_c, c> \ge 0$.

Since

$$|a| \ge \Omega'(|a_c| + |c|)\,|c|\,,$$

it follows that

$$|a_c| \le \frac{1}{\Omega'}\frac{|a|}{|c|}.$$

Now, $\alpha_0 - \beta_0$ differs from $|a|^2 - |b|^2$ by at most

$$2\,\|H(\omega)\| \lesssim d_\Delta^d < H >_{\{\Lambda \atop U\}},$$

and

$$|a|^2 - |b|^2 = -2\tau <c, b-a> -2 <a_c, b-a> - |b-a|^2\,.$$

Since $| <c, b-a> | \ge 1$ it follows that

$$\tau \lesssim |\alpha_0 - \beta_0| + |a_c|\Delta' + (\Delta')^2 + d_\Delta^d < H >_{\{\Lambda \atop U\}}\,.$$

If now $|\alpha_0 - \beta_0| \lesssim C_1\Delta'$ then $|a| \le |a_c| + |\tau\ c|$ is

$$\le \text{cte.}(|a_c|\,\Delta'\,|c| + C_1(\Delta')^2\,|c| + d_\Delta^d < H >_{\{\Lambda \atop U\}}\,|c|)$$

$$\leq \frac{1}{2}|a| + \text{cte.}(C_1(\Delta')^2\,|c| + d_\Delta^d <H>_{\{{\Lambda \atop U}\}}\,|c|).$$

Since a violates (19) this is impossible. Therefore $|\alpha_0 - \beta_0| \gtrsim C_1\Delta'$ and $(18)_\kappa$ holds.

Hence, we have proved that $(18)_\kappa$ holds for any

$$\begin{cases} (a,b) \in \bigcup_{0<|c|\lesssim\Omega^{d-1}} D_{\Omega'}(c) \\ (a,b) \in (17) \end{cases}$$

under the condition at ∞. Therefore $(18)_\kappa$ holds for any $(a,b) \in (17)$.

4. Proof of condition at ∞. Let c_1 be a primitive vector in $0 < |c_1| \lesssim \Omega^{d-1}$, and let G be the Töplitz limit $H(c_1)$. Then G verifies (16), $G(\omega)$ and $\partial_\omega G(\omega)$ are Töplitz at ∞ and

$$<G>_{\{{\Lambda \atop U}\}} \leq <H>_{\{{\Lambda \atop U}\}}.$$

Clearly $G(\omega)$ is Hermitian and, by Lemma 4.2, $G(\omega)$ and $\partial_\omega G(\omega)$ are block diagonal over $\mathcal{E}_\Delta$, i.e. $G(\omega)$ and $\partial_\omega G(\omega)$ are $\mathcal{NF}_\Delta$. Moreover G is Töplitz in the direction c_1,

$$G_{a+tc_1}^{b+tc_1} = G_a^b, \quad \forall a, b, tc_1.$$

We want to prove that G verifies $(18)_{\kappa_1}$ for all $(a,b) \in (17) + (21)_{c_1}$, i.e. for all

$$|a-b| \lesssim \Delta' \quad \text{and} \quad ([a]-[b]) \perp c_1.$$

Since G is Töplitz in the direction c_1 it is enough to show this for

$$(22) \qquad |<a, \frac{c_1}{|c_1|}>| \quad \text{and} \quad |<b, \frac{c_1}{|c_1|}>| \leq |c_1|.$$

But then all divisors are large except finitely many which we can treat as above. □

7. Function with Töplitz-Lipshitz property (d = 2)

7.1. **Töplitz structure of the quadratic differential.**

The quadratic differential

$$<\zeta, \frac{\partial^2}{\partial\zeta^2} f(0,\varphi,r)\zeta>$$

has the form

$$<\zeta, A\zeta> = \sum_{a,b\in\mathcal{L}} <\zeta_a, A_a^b \zeta_b>,$$

where $A : \mathcal{L}\times\mathcal{L} \to gl(2,\mathbb{R})$ is a $gl(2,\mathbb{R})$-valued matrix. It is uniquely determined by the symmetry condition

$${}^t\!A_a^b = A_b^a.$$

Its properties are best seen in the complex variables

$$({}^tCAC)_a^b = \begin{pmatrix} P_a^b & Q_a^b \\ Q_b & \bar P_a^b \end{pmatrix}.$$

Consider for example the Schrödinger equation with a cubic potential, i.e.

$$F(x,u,\bar u) = u^2 \bar u^2.$$

Then

$$P_{a_1}^{a_2} = \sum_{\substack{b_1,b_2\in\mathcal{A} \\ b_1+b_2=a_1+a_2}} 2\sqrt{r_{b_1} r_{b_2}} e^{-i(\varphi_{b_1}+\varphi_{b_2})}$$

and

$$Q_{a_2}^{b_2} = \sum_{\substack{a_1,b_1\in\mathcal{A} \\ a_1-b_1=a_2-b_2}} 8\sqrt{r_{a_1} r_{b_1}} e^{i(\varphi_{a_1}-\varphi_{b_1})}.$$

In particular

$$\begin{cases} P \text{ is symmetric} \\ Q \text{ is Hermitian.} \end{cases}$$

Moreover Q is Töplitz,

$$Q_{a+c}^{b+c} = Q_a^b \quad \forall a,b,c,$$

and (since $\mathcal{A}$ is finite) its elements are zero at finite distance from the diagonal. In particular, this matrix is Töplitz-Lipschitz and has exponential decay off the diagonal $a=b$. P is also Töplitz-Lipschitz with exponential decay but *in a different sense*:

$$P_{a+c}^{b-c} = P_a^b \quad \forall a,b,c,$$

and has exponential decay off the "anti-diagonal" $\{a=-b\}$.

7.2. **Töplitz-Lipschitz matrices $\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{R})$.** We consider the space $gl(2,\mathbb{C})$ of all complex 2×2-matrices provided with the scalar product

$$Tr({}^t\bar A B).$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and consider the orthogonal projection π of $gl(2,\mathbb{C})$ onto the subspace

$$M = \mathbb{C}I + \mathbb{C}J.$$

For a matrix

$$A : \mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$$

we define πA through

$$(\pi A)_a^b = \pi A_a^b, \quad \forall a,b.$$

We define the supremum-norms

$$|A|_\gamma^\pm = \sup_{(a,b)\in\mathcal{L}\times\mathcal{L}} |A_a^b| e^{\gamma|a\mp b|}$$

and

$$|A|_\gamma = \max(|\pi A|_\gamma^+, |A-\pi A|_\gamma^-).$$

A is said to have a Töplitz-limit at ∞ in the direction c if, for all a,b the two limits

$$\lim_{t\to+\infty} A_{a+tc}^{b\pm tc} \ \exists \ = A_a^b(\pm,c).$$

$A(\pm,c)$ are new matrices which are Töplitz/"anti-Töplitz" in the direction c, i.e.

$$A_{a+c}^{b+c}(+,c) = A_a^b(+,c) \quad \text{and} \quad A_{a+c}^{b-c}(-,c) = A_a^b(-,c).$$

If $|A|_\gamma < \infty$, $\gamma > 0$, then

$$\pi A(-,c) = (A-\pi A)(+,c) = 0.$$

We say that A is Töplitz at ∞ if all Töplitz-limits $A(\pm,c)$ exist.
We define the Lipschitz-constants

$$\mathrm{Lip}_{\Lambda,\gamma}^\pm A = \sup_{c\neq 0} \sup_{(a,b)\in D_\Lambda(c)} |(A-A(\pm,c))_a^{\pm b}| \max(\frac{|a|}{|c|},\frac{|b|}{|c|}) e^{\gamma|a\mp b|}$$

and the Lipschitz-norm

$$< A >_{\Lambda,\gamma} = \max(\mathrm{Lip}_{\Lambda,\gamma}^+ \pi A + |\pi A|_\gamma^+, \mathrm{Lip}_{\Lambda,\gamma}^-(I-\pi)A + |(I-\pi)A|_\gamma^-).$$

We say that A *Töplitz-Lipschitz* if $< A >_{\Lambda,\gamma} < \infty$ for some Λ,γ. The most important property is a product formula.

Proposition 7.1. *Let $A_1,\dots,A_n : \mathcal{L}\times\mathcal{L}\to\mathbb{C}$ be Töplitz-Lipschitz matrices with exponential decay off-diagonal, i.e.*

$$|A_j|_\gamma < \infty \quad j=1,\dots,n, \ \gamma>0.$$

Then $A_1\cdots A_n$ is Töplitz-Lipschitz and

$$< A_1\cdots A_n >_{\Lambda+6,\gamma'} \le$$
$$(\mathrm{cte.})^n \Lambda^2 (\tfrac{1}{\gamma-\gamma'})^{(n-1)d+1} [\textstyle\sum_{1\le k\le n} \prod_{\substack{1\le j\le n\\ j\neq k}} |A_j|_{\gamma_j} < A_k >_{\Lambda,\gamma_k}],$$

where all $\gamma_1,\dots,\gamma_n$ are $=\gamma$ except one which is $=\gamma'$.

7.3. **Functions with Töplitz-Lipschitz property.** Let $\mathcal{O}^\gamma(\sigma)$ be the set of vectors in the complex space $l^2_\gamma(\mathcal{L},\mathbb{C})$ of norm less than σ, i.e.

$$\mathcal{O}^\gamma(\sigma)=\{\zeta\in\mathbb{C}^{\mathcal{L}}\times\mathbb{C}^{\mathcal{L}}:\|\zeta\|_\gamma<\sigma\}.$$

Our functions $f:\mathcal{O}^0(\sigma)\to\mathbb{C}$ will be defined and real analytic on the domain $\mathcal{O}^0(\sigma)$. [4] Its first differential

$$l^2_0(\mathcal{L},\mathbb{C})\ni\hat\zeta\mapsto<\hat\zeta,\frac{\partial f}{\partial\zeta}(\zeta)>$$

defines a unique vector $\frac{\partial f}{\partial\zeta}(\zeta)$ in $l^2_0(\mathcal{L},\mathbb{C})$, and its second differential

$$l^2_0(\mathcal{L},\mathbb{C})\ni\hat\zeta\mapsto<\hat\zeta,\frac{\partial^2 f}{\partial\zeta^2}(\zeta)\hat\zeta>$$

defines a unique matrix $\frac{\partial^2 f}{\partial\zeta^2}(\zeta)$ $\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$ which is symmetric, i.e.

$${}^tA_a^b=A_b^a.$$

We say that f is *Töplitz at* ∞ if the matrix $\frac{\partial^2 f}{\partial\zeta^2}(\zeta)$ is Töplitz at ∞ for all $\zeta\in\mathcal{O}^0(\sigma)$. We define the norm

$$[f]_{\Lambda,\gamma,\sigma}$$

to be the smallest C such that

$$\begin{cases}|f(\zeta)|\le C & \forall\zeta\in\mathcal{O}^0(\sigma)\\ \|\partial_\zeta f(\zeta)\|_{\gamma'}\le\frac{1}{\sigma}C & \forall\zeta\in\mathcal{O}^{\gamma'}(\sigma),\ \forall\gamma'\le\gamma,\\ <\partial^2_\zeta f(\zeta)>_{\Lambda,\gamma'}\le\frac{1}{\sigma^2}C & \forall\zeta\in\mathcal{O}^{\gamma'}(\sigma),\ \forall\gamma'\le\gamma.\end{cases}$$

7.4. **A short remark on the proof of Theorem A.** Our Hamiltonians are functions of $\zeta=(\xi,\eta),r,\varphi$ and ω. We measure these functions in a norm given by

- the $[\ \ ]_{\Lambda,\gamma,\sigma}$-norm for ζ
- the sup-norm over a complex domain $|r|<\mu$ and $|\Im\varphi|<\rho$
- the $\mathcal{C}^1$-norm in ω.

In this norm we estimate the solution s,k of the homological equation (8) (described in Section 3.3) and the transformed Hamiltonian

$$h'+f'=(h+f)\circ\Phi^1,$$

where Φ^1 is the time-one-map of the Hamiltonian vector field of s,

In order to carry this out we study the behavior of this norm under truncations, Poisson brackets, flows and compositions.

[4]The space $l^2_\gamma(\mathcal{L},\mathbb{C})$ is the complexification of the space $l^2_\gamma(\mathcal{L},\mathbb{R})$ of real sequences. "real analytic" means that it is a holomorphic function which is real on $\mathcal{O}^0(\sigma)\cap l^2_\gamma(\mathcal{L},\mathbb{R})$.

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University of Paris 7, Department of Mathematics, Case 7052, 2 place Jussieu, Paris, France

E-mail address: `hakane@math.jussieu.se`

Heriot-Watt University, Department of Mathematics, Edinburgh

E-mail address: `Kuksin@math.hw.ac.uk`