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KAM theory with application to nonlinear wave equations

Xiaoping Yuan
School of Mathematical Sciences
Fudan University
Shanghai 200433
China
xpyuanfudan.edu.cn; yuanxiaopinghotmail.com

Abstract

1. Introduction
Consider nonlinear wave equation (NLW):

$$u_{tt} - u_{xx} + V(x)u + \text{h.o.t.} = 0, \quad u(t,0) = u(t,\pi) = 0. \tag{1}$$

Robinowitz showed that there was a periodic solution of period T with $T/\pi \in \mathbb{Q}$. Let us begin with the following two questions.

1. A natural problem is that what happens when $T/\pi \in \mathbb{R}\backslash\mathbb{Q}$.
2. In geometric view of point, the periodic solution can be regarded as an invariant closed curve, or invariant torus of Dimension 1. Therefore, *another question is whether or not there is a or many invariant tori of dimension N with* $N > 1$. KAM theory of infinite dimension can answer these questions. In the first talk, I would like to introduce briefly the KAM theorem by Kuksin, Wayne, Pöschel where the normal frequencies of Hamiltonian are simple. Then apply the theorem to (1) with all cases: $V(x)$ depending on some parameters or being a random potential (Kuksin, Wayne); $V(x) \equiv m \neq 0$ (Bobenko-Kuksin, Pöschel); $V(x)$ being prescribed; $V(x) \equiv 0$.

The KAM theorem mentioned above requires the so-called second Melnikov conditions which prevent its application to higher spatial dimensional NLE equations. In my following talks, I will present a new KAM theorem which can apply to many PDEs. KAM theory are usually figured to be very complicated and tedious, although it is

a powerful tool. I hope my talks are not too technical and difficult. Anyway, the basic idea of KAM theory is quite simple.

2. Linear equations

In this section, our purpose is to find quasi-periodic solutions of a linear system of variable coefficients

$$\dot{x} + \big(\lambda + \varepsilon B(t)\big)x = f(t), \quad x \in \mathbb{R}^m$$

where $B(t)$, $f(t)$ are quasi-periodic with frequency vector $\omega \in \mathbb{R}^n$. In solving the linear system, we need to find the inverse of a "big" matrix A, here arising the small divisor problem. We will use the variational technique of eigenvalues instead of Fröhlich-Spencer one to overcome the small divisor problem. This makes our technique very simple.

3. Derivation of homological equations

KAM theory announces the most invariant tori of an integrable Hamiltonian persist under a small perturbation. The preserved tori can be constructed by solving a series of linear equation systems. In this section, we derive those linear equation systems and give out their solutions by using the results in the previous section.

4. Iteration steps and application to $d > 1$ dim PDEs

Every solution of the linear equation systems can produce a symplectic transformation. In this section we will show that the composition of all symplectic transformations, denoting it by Ψ, change the Hamiltonian

$$H = (\omega, y) + \langle \Lambda u, u\rangle + \varepsilon H_1(x, y, u)$$

into

$$H \circ \Psi = (\widetilde{\omega}, y) + \langle A(x)u, u\rangle + O\big(|y|^2 + |y|\,\|u\| + \|u\|^2)\big)$$

for most ω, provided that $\Lambda = \mathrm{diag}(\lambda_j : j \in \mathbb{Z}^d)$ and

$$\lambda_j \approx |j|^\kappa, \quad \kappa > 0.$$

Therefore, $\Psi^{-1}(\mathbb{T}^n \times \{y = 0\} \times \{u = 0\})$ is an invariant tori of H, on which any motion is quasi-periodic. Finally, we apply the obtained results to the nonlinear wave equation of higher dimension.

References

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