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On the periodic KdV-equation in weighted Sobolev spaces

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Abstract

We discuss the initial value problem for the periodic KdV equation,

$$u_t = -u_{xxx} + 6uu_x, \quad u|_{t=0} = u_0, \tag{1}$$

where all functions are considered to be defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. According to one of the first results in this direction due to Bona and Smith [1] this initial value problem is globally well-posed on $\mathcal{H}^m = H^m(\mathbb{T}, \mathbb{R})$ with $m \geq 2$ in the sense of Hadamard: solutions exist for all time, are unique, and depend continuously on their initial values.

Here we will discuss *high regularity* solutions. These are solutions in a general class of weighted Sobolev spaces $\mathcal{H}^w$ within $\mathcal{H}^0$, that encompass analytic and Gevrey spaces, among others, as well as the spaces $\mathcal{H}^m$. One of the main results is the following.

Theorem *The periodic* KdV *equation is globally uniformly well-posed in every space* $\mathcal{H}^w$ *with a subexponential weight* w*. That is, for each initial value* u *in* $\mathcal{H}^w$ *the associated Cauchy problem has a global solution in* $\mathcal{H}^w$ *, giving rise to a continuous flow* $\mathbb{R} \times \mathcal{H}^w \to \mathcal{H}^w$ *which is even uniformly continuous on bounded subsets of* $\mathcal{H}^w$*.*

These results are based on two observations. First, the periodic KdV equation is an *infinite-dimensional, integrable Hamiltonian system*, which even admits global Birkhoff coordinates $(x_n, y_n)_{n\geq 1}$. Second, the KdV flow defines an isospectral deformation among potentials of Hill operator $-d^2/dx^2 + u$, and the spectral asymptotics of u are closely connected with the asymptotics of (x_n, y_n) on one hand, and spectral asymptotics on the other hand. For a comprehensive list of references see in particular [2].

References

[1] J.L.Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A **27** (1975), no. 8, 555–601.

[2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Local and global well-posedness for non-linear dispersive and wave equations*. `www.math.ucla.edu/ tao/Dispersive`.

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