On the Periodic KdV Equation in Weighted Sobolev Spaces

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1 Results

Well Posedness of KdV

Initial value problem for KdV equation

$$u_t = -u_{xxx} + 6uu_x, \qquad u\big|_{t=0} = u_0,$$

with periodic boundary conditions: $\mathcal{H}^r = H^r(\mathbb{T}, \mathbb{R})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Notions

»Globally well posed in $\mathcal{H}^r$« global solution map

$$\mathbb{R} \times \mathcal{H}^r \to \mathcal{H}^r, \quad (t, u_0) \mapsto \varphi(t, u_0)$$

exists and is continuous.

»Globally uniformly well posed in $\mathcal{H}^r$« solution map is uniformly continuous on “compact $\times$ bounded” subsets of $\mathbb{R} \times \mathcal{H}^r$.

Known Results

First results – in 70’s

Temam; Sjöberg; Bona & Smith : in Sobolev spaces $\mathcal{H}^r$, $r = 2, \ldots$

Low regularity solutions – since 90’s

Bourgain; Kenig, Ponce & Vega; Colliander, Keel, Staffilani, Takaoka & Tao; Kappeler & Topalov :

KdV globally well posed in $\mathcal{H}^r$ for $r \geqslant -1$, and uniformly so for $r \geqslant -1/2$.

High regularity solutions – since 00’s

Bona; Grujic & Kalisch : in Gevrey spaces, analytic spaces, *weighted Sobolev spaces* . . .

Weighted Sobolev Spaces

Weights

$$w: \ \mathbb{Z} \to \mathbb{R}, \quad n \mapsto w_n$$

symmetric, normalized, and submultiplicative:

$$w_n \geqslant 1, \qquad w_{-n} = w_n, \qquad w_{n+m} \leqslant w_n w_m.$$

Norm

$$\|u\|_w^2 = \sum_{n\in\mathbb{Z}} w_n^2 |u_n|^2, \qquad u = \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n x}.$$

Spaces

$$\mathcal{H}^w = \{u \in L^2 : \|u\|_w < \infty\}.$$

Examples of Weighted Sobolev Spaces

With

$$\langle n \rangle = 1 + |n|, \qquad r \geqslant 0, \quad a > 0, \quad 0 < \sigma < 1$$

then

- $w_n = \langle n \rangle^r$: *Sobolev* spaces $\mathcal{H}^r = H^r(\mathbb{T}, \mathbb{R})$
- $w_n = \langle n \rangle^r \mathrm{e}^{a|n|}$: *Abel* spaces $\mathcal{H}^{r,a}$
- $w_n = \langle n \rangle^r \mathrm{e}^{a|n|^\sigma}$: *Gevrey* spaces $\mathcal{H}^{r,a,\sigma}$

Note

$$\mathcal{H}^{r,a} = \mathcal{H}^{r,a,1} \subsetneq \mathcal{H}^{r,a,\sigma} \subsetneq \mathcal{H}^{r,a,0} = \mathcal{H}^r.$$

Results

Theorem 1 *Periodic KdV is globally uniformly well posed in*

- *Sobolev spaces* $\mathcal{H}^r$, $r \geqslant 1$,
- *Gevrey spaces* $\mathcal{H}^{r,a,\sigma}$, $0 < \sigma < 1$,
- *every space* $\mathcal{H}^w$ *with strictly subexponential weight* w. ⋊

Subexponential:

$$\lim_{n\to\infty} \frac{\log w_n}{n} = 0, \qquad \frac{\log w_n}{n} \searrow 0 \quad \text{eventually}$$

That is: for each inital value u_0 in such a $\mathcal{H}^w$ the Cauchy problem has a global solution in $\mathcal{H}^w$, and the associated flow is uniformly continuous on bounded subset of $\mathcal{H}^w$.

More Results

Theorem 2 *Periodic KdV is "almost" globally uniformly well posed in ever space* $\mathcal{H}^w$ *with an exponential weight* w. ⋊

That is: for each bounded subset $\mathcal{B} \subset \mathcal{H}^w$ there is $0 < \rho \leqslant 1$ so that we have a continuous flow

$$\mathbb{R} \times \mathcal{B} \to \mathcal{H}^{w^\rho}, \quad (t, u_0) \mapsto \varphi^t(u_0).$$

Theorem 3 *These well posedness results also hold for every other KdV equation in the KdV hierarchy.* ⋊

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Step 1: Birkhoff Coordinates

■ KdV as Hamiltonian System

Hamiltonian

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H_{\mathrm{KdV}}}{\partial u}, \qquad H_{\mathrm{KdV}} = \int_{\mathbb{T}} \left(\tfrac{1}{2}u_x^2 + u^3\right) \mathrm{d}x$$

Phase space

$$\mathcal{H}_0^m = H^m(\mathbb{T},\mathbb{R}) \cap \left\{\textstyle\int_{\mathbb{T}} u = 0\right\}, \qquad m \geqslant 1.$$

Poisson bracket

$$\{F,G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial u}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G}{\partial u}\,\mathrm{d}x,$$

nondegenerate on $\mathcal{H}_0^1$. Then

$$u_t = \{u, H_{\mathrm{KdV}}\}.$$

■ KdV as Integrable System

Well known

KdV is *integrable* Hamiltonian system. Indeed, it admits

Birkhoff coordinates

that is, cartesian action angle coordinates.

Sequence spaces

$$h_*^w = \{z = (x_n, y_n)_{n\geqslant 1} : \|z\|_{w_*} < \infty\},$$

where

$$\|z\|_{w_*}^2 = \sum_{n\geqslant 1} n w_n^2 \left(|x_n|^2 + |y_n|^2\right).$$

Write h_*^m for Sobolev weights $w_n = n^m$.

■ Birkhoff coordinates

Theorem A [*Kappeler & P*] *There exists a bi-analytic, symplectic diffeo*

$$\Omega : \mathcal{H}_0^0 \to h_*^0,$$

such that KdV-Hamiltonian becomes function of actions:

$$H_{KdV} \circ \Omega^{-1} = H(I_1, I_2, \ldots), \qquad I_n = \tfrac{1}{2}(x_n^2 + y_n^2).$$

Moreover, for each $m \geqslant 1$*,*

$$\Omega : \mathcal{H}_0^m \to h_*^m. \qquad \rtimes$$

symplectic: Gardner bracket $\mapsto$ standard bracket.

Using Birkhoff coordinates ...

Invariant tori Every orbit of KdV sits on *invariant torus:*

$$T_I = \prod_{n\geqslant 1} S_{I_n} = \prod_{n\geqslant 1} \{x_n^2 + y_n^2 = 2I_n\},$$

circling each circle with *constant speed:*

$$\psi^t: \quad \dot I_n = 0, \quad \dot\theta_n = \omega_n = H_{I_n}(I_1, I_2, \dots).$$

KdV well-posed in $\mathcal{H}_0^m$

$$\begin{array}{ccc} u \in \mathcal{H}_0^m & \xrightarrow{\ \Omega\ } & (x,y) \in h_*^m \\ \phi^t \downarrow & & \downarrow \psi^t \\ \phi^t(u) \in \mathcal{H}_0^m & \xleftarrow{\ \Omega^{-1}\ } & \psi^t(x,y) \in h_*^m \end{array}$$

Extension 1 for Ω: $\mathcal{H}_0^0 \to h_*^0$

Theorem B *For each strictly subexponential weight w,*

$$\Omega\Big|_{\mathcal{H}_0^w}: \ \mathcal{H}_0^w \to h_*^w$$

is a bi-analytic diffeomorphism onto. ⋊

Corollary *KdV is well posed in $\mathcal{H}^w$, w strictly subexponential.* ⋊

$$\begin{array}{ccc} u \in \mathcal{H}_0^w & \xrightarrow{\ \Omega\ } & (x,y) \in h_*^w \\ \phi^t \downarrow & & \downarrow \psi^t \\ \phi^t(u) \in \mathcal{H}_0^w & \xleftarrow{\ \Omega^{-1}\ } & \psi^t(x,y) \in h_*^w \end{array}$$

Extension 2 for Ω: $\mathcal{H}_0^0 \to h_*^0$

Theorem C *Given an exponential weight w, for each bounded subset $B \subset h_*^w$ there is $0 < \rho \leqslant 1$ so that*

$$\Omega^{-1}(B) \subset \mathcal{H}_0^{w^\rho}. \quad ⋊$$

For example, for $w_n = \mathrm{e}^{an}$,

$$(w^\rho)_n = w_n^\rho = \mathrm{e}^{\rho a n}.$$

Corollary *KdV is »almost« well posed in $\mathcal{H}^w$, when w is exponential.* ⋊

3 Step 2: Regularity

■ **Line of argument**

Theorem B,

$$\Omega|_{\mathcal{H}_0^w} : \ \mathcal{H}_0^w \to h_*^w,$$

and Theorem C based on two observations:

$$\begin{aligned} &\text{asymptotics of Birkhoff co's } (x,y) \text{ of } u \\ \rightsquigarrow\ &\text{asymptotics of } \textit{spectral gaps } \gamma_n \text{ of } u \\ \rightsquigarrow\ &\text{regularity of } u \end{aligned}$$

■ **Spectral Gaps of u**

Hill's equation with potential u

$$-y'' + uy = \lambda y, \qquad y(x+2) = y(x).$$

periodic spectrum

$$\lambda_0(u) < \lambda_1^-(u) \leqslant \lambda_1^+(u) < \lambda_2^-(u) \leqslant \dots,$$

spectral gaps

$$\gamma_n(u) = \lambda_n^+(u) - \lambda_n^-(u).$$

Reminder KdV flow is isospectral deformation:

$$\gamma_n(\varphi^t(u)) = \gamma_n(u), \qquad t \in \mathbb{R}.$$

■ **Asymptotics of Birkhoff co's**

Fact about actions of KdV

$$0 \neq 8\pi \frac{nI_n}{\gamma_n^2} \sim 1, \qquad I_n = \tfrac{1}{2}(x_n^2 + y_n^2).$$

Hence

$$nI_n \sim n(x_n^2 + y_n^2) \sim \gamma_n^2,$$

and thus

$$\begin{aligned} &\text{asymptotics of Birkhoff co's } (x,y) \text{ of } u \\ \rightsquigarrow\ &\text{asymptotics of spectral gaps } \gamma_n \text{ of } u \end{aligned}$$

... at least in the *real case.*

■ **Asymptotics of spectral gaps, I**

Theorem I [***Marčenko & Ostrowskiĭ, ..., P***] *For subexponential or exponential weight w:*

$$u \in \mathcal{H}^w \ \Rightarrow\ (\gamma_n(u)) \in h^w. \qquad ⋊$$

Corollary *Birkhoff map Ω maps $\mathcal{H}_0^w$ into h_*^w.* ⋊

Because

$$\begin{aligned} u \in \mathcal{H}^w \ &\Rightarrow\ (\gamma_n) \in h^w \\ &\Rightarrow\ (nI_n) \in h^w \\ &\Rightarrow\ (x_n, y_n) \in h_*^w \end{aligned}$$

Asymptotics of spectral gaps, II

Theorem II [*Diakov & Mityagin, P*] *For strictly subexponential weight w:*

$$(\gamma_n(u)) \in h^w \;\Rightarrow\; u \in \mathcal{H}^w. \quad \rtimes$$

Corollary *Birkhoff map Ω maps $\mathcal{H}_0^w$ onto h_*^w.* $\rtimes$

Because

$$\begin{aligned} &(x,y) \in h_*^w \subset h_*^0 \\ \Rightarrow\quad & u = \Omega^{-1}(x,y) \in \mathcal{H}_0^0 \quad \text{with} \quad \gamma_n^2 \sim n(x_n^2 + y_n^2) \\ \Rightarrow\quad & (\gamma_n(u)) \in h^w \\ \Rightarrow\quad & u \in \mathcal{H}^w \end{aligned}$$

Summary of Subexponential Case

Summary *Let w be a strictly subexponential weight, such as Gevrey or Sobolev. Then the Birkhoff coordinate map Ω maps $\mathcal{H}_0^w$ into and onto h_*^w.* $\rtimes$

In fact,

$$\Omega\colon\; \mathcal{H}_0^w \to h_*^w$$

is a *bi-analytic* diffeomorphism.

But this requires to consider complex potentials, too …

Problem in Exponential Case

Problem For instance for $w_n = \langle n\rangle^r \mathrm{e}^{an}$:

$$(\gamma_n(u)) \in h^{r,a} \quad \not\Rightarrow \quad u \in \mathcal{H}^{r,a}.$$

Example Finite gap potentials with poles: $\gamma_n = 0$, $n \geqslant N \;\Rightarrow$

$$(\gamma_n(u)) \in h^{r,a} \qquad \text{for } all\ a > 0,$$

but u *not* in $\mathcal{H}^{r,a}$ for *all* $a > 0$.

Result in Exponential Case

Theorem III [*Trubowitz; P*] *For exponential weight w:*

$$(\gamma_n(u)) \in h^w \;\Rightarrow\; u \in \mathcal{H}^{w^\rho},$$

where $0 < \rho \leqslant 1$ depends on $\|u\|_{L^2}$ and $\sum_n w_n^2 \gamma_n^2$. $\rtimes$

Corollary

$$B \subset h_*^w \text{ bounded} \;\Rightarrow\; \Omega^{-1}(B) \subset \mathcal{H}_0^{w^\rho}, \quad 0 < \rho \leqslant 1. \qquad \rtimes$$

Corollary *For each bounded subset $\mathcal{B} \subset \mathcal{H}_0^w$ there is $0 < \rho \leqslant 1$ such that there is a KdV flow*

$$\mathbb{R} \times \mathcal{B} \to \mathcal{H}_0^{w^\rho}, \quad (t, u_0) \mapsto \varphi^t(u_0). \qquad \rtimes$$

4 Application

Perturbed KdV

Perturbation

$$H = H_{\mathrm{KdV}} + \varepsilon K, \qquad X_K = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial K}{\partial u} \in \mathcal{H}^w.$$

KdV well-posed in $\mathcal{H}^w$ $\Rightarrow$
perturbation problem entirely within $\mathcal{H}^w$.

KAM theory within $\mathcal{H}^w$

In the proof in »KdV & KAM« by Kappeler & P, replace weights

$$\mathrm{e}^{a|n|} \qquad \text{by} \qquad w_n.$$

This answers a question raised by Jürgen Moser many years ago.

5 The End