Hill's Potentials in Weighted Sobolev Spaces and their Spectral Gaps

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1 Results

Hill's equation

Hill's operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q \qquad \text{on} \qquad L^2[0,1]$$

with periodic or anti-periodic bc and *real* L^2-potential q.

Spectrum Real, pure point and unbounded:

$$\lambda_0(q) < \lambda_1^-(q) \leqslant \lambda_1^+(q) < \lambda_2^-(q) \leqslant \dots,$$

asymptotic behaviour

$$\lambda_n^\pm(q) = n^2\pi^2 + [q] + \ell_n^2.$$

Gap lengths

$$\gamma_n(q) = \lambda_n^+(q) - \lambda_n^-(q) = \ell_n^2.$$

Gap lengths ⇝ Regularity of the potential ?

Classic answers *Hochstadt* (1963)

$$q \in C^\infty(S^1,\mathbb{R}) \iff \gamma_n(q) = O(n^{-k}) \text{ for all } k \geqslant 0.$$

Marčenko & Ostrowskiĭ (1975)

$$q \in H^k(S^1,\mathbb{R}) \iff \sum_{n\geqslant 1} n^{2k}\gamma_n^2(q) < \infty.$$

Trubowitz (1977)

$$q \in C^\omega(S^1,\mathbb{R}) \iff \gamma_n(q) = O(\mathrm{e}^{-an}) \text{ for some } a > 0.$$

More recent answers *Sansuc & Tkachenko (96), Kappeler & Mityagin (99), Djakov & Mityagin (02,03)*

for Gevrey & complex potentials, weighted Sobolev spaces ...

Weighted Sobolev Spaces

Weights

$$w : \mathbb{Z} \to \mathbb{R}, \quad n \mapsto w_n$$

symmetric, normalized, and submultiplicative:

$$w_n \geqslant 1, \qquad w_{-n} = w_n, \qquad w_{n+m} \leqslant w_n w_m.$$

Norm

$$\|u\|_w^2 = \sum_{n\in\mathbb{Z}} w_n^2 |u_n|^2, \qquad u = \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n x}.$$

Spaces

$$\mathcal{H}^w = \{u \in L^2 : \|u\|_w < \infty\},$$
$$h^w = \{u \in \ell^2 : \|u\|_w < \infty\}.$$

Examples of Weighted Sobolev Spaces

With

$$\langle n\rangle = 1 + |n|, \qquad r \geqslant 0, \quad a > 0, \quad 0 < \sigma < 1$$

let

- $w_n = \langle n\rangle^r$: *Sobolev* spaces $\mathcal{H}^r$
- $w_n = \langle n\rangle^r \mathrm{e}^{a|n|}$: *Abel* spaces $\mathcal{H}^{r,a}$
- $w_n = \langle n\rangle^r \mathrm{e}^{a|n|^\sigma}$: *Gevrey* spaces $\mathcal{H}^{r,a,\sigma}$
- $w_n = \langle n\rangle^r \exp\left(\dfrac{a|n|}{1+\log^\alpha\langle n\rangle}\right), \qquad \alpha > 0$

Note

$$\mathcal{H}^{r,a} = \mathcal{H}^{r,a,1} \subsetneq \mathcal{H}^{r,a,\sigma} \subsetneq \mathcal{H}^{r,a,0} = \mathcal{H}^r.$$

The Forward Result

Forward Theorem [*Kappeler & Mityagin, P*]
For any submultiplicative weight w,

$$q \in \mathcal{H}^w \Rightarrow \gamma(q) \in h^w. \qquad ⋊$$

Detailed estimate
For $q \in \mathcal{H}^w$,

$$\sum_{n\geqslant N} w_n^2 |\gamma_n(q)|^2 \leqslant 9\|T_N q\|_w^2 + \frac{576}{N}\|q\|_w^4$$

for $N \geqslant 4\|q\|_w$, where $T_N q = \sum_{|n|\geqslant N} q_n \mathrm{e}^{2n\pi i x}$.

Converse Results

No one-to-one converse for *exponential* weights:

$$\lim_{n\to\infty} \frac{\log w(n)}{n} > 0,$$

because finite-gap potentials need not be entire functions.

One-to-one converse for *strictly subexponential* weights:

$$\lim_{n\to\infty} \frac{\log w(n)}{n} = 0$$

and

$$\frac{\log w(n)}{n} \searrow 0 \quad \text{eventually}$$

Examples: all non-exponential weights above

Converse Results ...

Converse Theorem *Suppose $q \in \mathcal{H}^o$ is real and*

$$\gamma(q) \in h^w.$$

If w is strictly subexponential, then $q \in \mathcal{H}^w$.
If w is exponential, then q is real analytic. ⋊

Corollary *If q is real and w strictly subexponential, then*

$$q \in \mathcal{H}^w \quad \Leftrightarrow \quad \gamma(q) \in h^w. \qquad ⋊$$

2 Reduction

Fourier Block Decomposition [*Kappeler & Mityagin*]

Hill's equation $-y'' + qy = \lambda y$ for $\lambda \sim n^2\pi^2$ *large*:

$$-y'' \sim n^2\pi^2 y \quad \rightsquigarrow \quad y \sim \mathrm{e}^{\pm n\pi i x}.$$

Splitting

$$\begin{aligned} H^w &= \mathcal{P}_n \oplus \mathcal{Q}_n \\ &= \operatorname{span}\{\mathrm{e}^{\pm n\pi i x}\} \oplus \operatorname{span}\{\mathrm{e}^{\pm k\pi i x} : |k| \neq n\}, \end{aligned}$$

and

$$y = u + v, \qquad u \in \mathcal{P}_n, \quad v \in \mathcal{Q}_n.$$

Decomposition of $y'' + \lambda y = qy$

(P) $\quad A_\lambda u = P_n V(u+v), \qquad A_\lambda = D^2 + \lambda,$

(Q) $\quad A_\lambda v = Q_n V(u+v), \qquad V = q\,\cdot$

- **Q-Equation** $A_\lambda v = Q_n V(u + v)$

  For $\lambda \sim n^2\pi^2$, A_λ has compact inverse on

  $$\mathcal{Q}_n = \operatorname{span}\{e^{\pm k\pi i x} : |k| \neq n\}.$$

Lemma *For $q \in H^w$ and $\lambda \in U_n = \{|\lambda - n^2\pi^2| < 12n\}$,*

$$\|A_\lambda^{-1} Q_n V\|_w \leqslant \frac{2}{n}\|q\|_w. \qquad ⋊$$

Solution of Q-equation

$$v = A_\lambda^{-1} Q_n V u + \dots, \qquad n \geqslant 4\|q\|_w.$$

- **P-Equation** $A_\lambda u = P_n V(u + v)$

$$A_\lambda u = P_n W_n u, \qquad W_n = V + V A_\lambda^{-1} Q_n V + \dots ,$$

or

$$S_n u = 0, \qquad S_n = A_\lambda - P_n W_n.$$

Two-dimensional equation:

$$S_n = \begin{pmatrix} \lambda - \sigma_n - a_n & c_n \\ c_{-n} & \lambda - \sigma_n - a_n \end{pmatrix}, \qquad \sigma_n = n^2\pi^2.$$

Lemma λ *periodic eigenvalue near σ_n iff* $\det S_n = 0$. ⋊

Estimates For $q \in H^w$ and $\lambda \in U_n = \{|\lambda - n^2\pi^2| < 12n\}$,

$$|a_n - q_0|,\ w_n|c_n - q_n| \leqslant \frac{4}{n}\|q\|_w^2.$$

3 Proofs

- **Forward Problem: Gap Estimates**

Reduced system

$$S_n = \begin{pmatrix} \lambda - \sigma_n - a_n & c_n \\ c_{-n} & \lambda - \sigma_n - a_n \end{pmatrix}, \qquad \sigma_n = n^2\pi^2.$$

Determinant

$$\det S_n = (\lambda - \sigma_n - a_n)^2 - |c_n|^2,$$

has two real roots $\rho_{1,2}$ with distance $\lessdot |c_n|$.

Result

$$|\gamma_n| = |\rho_1 - \rho_2| \lessdot |c_n| \lessdot q_n + \frac{\|q\|_w^2}{n w_n}.$$

This proves Forward Theorem.

Converse Problem: Easy Part

Geometric step Starting with gap lengths,

$$|c_n| < |\gamma_n|, \qquad n \gg 1.$$

Analytic step

$$c_n = q_n + O_2(\dots, q_k, \dots).$$

This is a

- nonlinear system in Fourier coefficients of q,
- provides bound of q_n in terms of c_n
- and thus γ_n.

See *Mityagin et al.*

Adapted Fourier Coefficients

Diagonal of

$$S_n = \begin{pmatrix} \lambda - \sigma_n - a_n & c_n \\ c_{-n} & \lambda - \sigma_n - a_n \end{pmatrix}, \qquad \sigma_n = n^2\pi^2,$$

vanishes at unique point $\alpha_n(q) = \sigma_n + a_n + \dots$.

Define *adapted Fourier coefficients*

(A) $$p_n = c_n(\alpha_n(q), q) = q_n + \dots$$

Observation For $n > ||q||_{L^2}$, the quantity p_n

- is defined for $q \in L^2$, not only $q \in H^w$,
- is *analytic function* of $q \in L^2$,
- inherits »regularity« of q in view of (A).

Fourier Coefficient Map

On

$$B_m^o = \left\{ q \in L^2 : ||q||_o < m \right\} \quad \ni \quad q = \sum_n q_n e^{n\pi i x}$$

define

$$\Phi_m : \; B_m^o \to L^2$$

by replacing q_n by p_n for $|n| > m$.

Proposition *Φ_m is a near identity diffeo on B_m^o, such that for any submultiplicative weight w,*

$$\Phi_m \big|_{B_m^w}, \qquad B_m^w = \{ q \in H^w : ||q||_w < m \} \subset B_m^o$$

is also a real analytic diffeo onto its image in H^w, which contains $B_{m/2}^w$. ⋊

»Abstract« Regularity Argument

Given $q \in L^2$ with asymptotics of its γ_n. Then we know:

$$\gamma_n \asymp c_n \asymp p_n, \qquad n \gg 1.$$

Choosing $m \geqslant 8||q||_o$ we then have $q \in B_m^o$ and

$$p = \Phi_m(q) \in H^w \subset L^2.$$

If we also *had*

(B) $$||p||_w \leqslant m/2, \quad \textit{i.e.} \quad \Phi_m(q) \in B_{m/2}^w \subset \Phi_m(B_m^w),$$

then, by inverse function theorem,

$$q = \Phi_m^{-1}(p) \in H^w, \qquad \text{QED.}$$

So we need estimate (B), an a priori bound ...

Getting Estimate for $p = \Phi_m(q)$

Want

$$\|p\|_w < m/2.$$

Have

$$p \in H^w \quad \Rightarrow \quad \|p\|_w < \infty,$$
$$p \in B^o_{m/6} \quad \Rightarrow \quad \|p\|_o < m/6,$$

by choosing m appropriately.

Idea Modify the weight w to w_ε, so that

- asymptotics not affected,
- lower order terms almost »no weight«, as in L^2.

Modified Weights, Subexponential Case

Choose

$$w_{\varepsilon,n} = w_n \wedge e^{\varepsilon|n|} \leqslant \begin{cases} w_n, & n \text{ large} \\ e^{\varepsilon|n|}, & n \text{ small} \end{cases}$$

Choosing N *large* and then ε *small:*

$$\|T_N p\|_{w_\varepsilon} \leqslant \|T_N p\|_w \leqslant \|p\|_0,$$
$$\|p - T_N p\|_{w_\varepsilon} \leqslant 2\|p\|_0.$$

Result

$$\|p\|_{w_\varepsilon} \leqslant 3\|p\|_o \leqslant m/2.$$

Conclusion $q = \Phi_m^{-1}(p) \in H^{w_\varepsilon} = H^w$.

Modified Weights, Exponential Case

Choose again

$$w_{\varepsilon,n} = w_n \wedge e^{\varepsilon|n|}$$

But this time, for ε sufficiently small,

$$w_{\varepsilon,n} = e^{\varepsilon|n|}, \quad \text{all } n$$

Conclusion

$$q = \Phi_m^{-1}(p) \in H^{w_\varepsilon} = H^{0,\varepsilon} \supsetneq H^w.$$

Complex Potentials

For complex L^2-potential q

- Hill's operator no longer self-adjoint,
- but same spectral asymptotics.

Lexicographic ordering

$$\lambda_0(q) \prec \lambda_1^-(q) \preccurlyeq \lambda_1^+(q) \prec \cdots \prec \lambda_n^-(q) \preccurlyeq \lambda_n^+(q) \prec \cdots.$$

Gap lengths

$$\gamma_n(q) = \lambda_n^+(q) - \lambda_n^-(q) = \ell_n^2$$

now complex valued.

But no longer sufficient for regularity results ...

Complex Potentials ...

Gasymov: *any* L^2-potential of the form

$$q = \sum_{n\geqslant 1} q_n e^{2n\pi i x} = \sum_{n\geqslant 1} q_n z^n \Big|_{z=e^{2\pi i x}}$$

is a *0-gap potential.*

In the complex case, the gap sequence therefore need not contain *any* information about the regularity of the potential.

Additional spectral data are required ...

Additional Spectral Data

Sansuc & Tkachenko: Consider size of *spectral triangles*

$$\Gamma_n = |\gamma_n| + |\delta_n|,$$

where for example

$$\delta_n = \mu_n - \tau_n = \mu_n - \frac{\lambda_n^+ + \lambda_n^-}{2}.$$

Converse Theorem II *Suppose $q \in \mathcal{H}^o$ is real or complex and*

$$\Gamma(q) \in h^w.$$

If w is strictly subexponential, then $q \in \mathcal{H}^w$.
If w is exponential, then q is real analytic. ⋊

4 The End