NATO Tutorial series

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1 Lecture 1

1.1 Syllabus

We will study how to derive cutting planes for integer programs from single constraint relaxations using the mixed-integer rounding (MIR) inequality, and its extensions.

1. Lecture 1: MIR inequality and extensions
2. Lecture 2: Gomory's master cyclic group polyhedron
3. Lecture 3: Group cuts
4. Lecture 4: MIR closure
5. Lecture 5: MIR cutting-plane proofs

1.2 Introduction

We will study mixed-integer programs given in the form

$$\min\{ex + fv \ : \ (x, v) \in P\} \text{ where}$$

$$P = \{(x, v) \in R^m : Ax + Cv = d, \quad x, v \geq 0, \ x \text{ integer}\},$$

possibly after slack variables are added. We denote the continuous relaxation of P by P^{LP}. We assume that all numerical data is rational. Let

$$Q \ = \ \Big\{v \in R^{|J|}, x \in Z^{|I|} \ : \ \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i \, x_i = b, \quad x, v \geq 0, \ x \text{ integer}\Big\},$$

where the single equation defining the set Q, called the *base* equation, is obtained by taking a linear combination of the equations defining P. In other words, $a = \lambda^T A$, $c = \lambda^T C$ and

$b = \lambda^T d$ for some real vector λ of appropriate dimension. As $P \subseteq Q$, valid inequalities for Q yield valid inequalities for P.

1.3 Basic mixed integer inequality

The MIR procedure provides a unifying framework to derive valid inequalities for mixed-integer sets. See Marchand and Wolsey[12] for examples. We now discuss this procedure. For a number v and an integer t, define $\hat{v} = v - \lfloor v \rfloor$, and $\hat{v}^t = tv - \lfloor tv \rfloor$. Define

$$Q^1 = \{ v \in R,\ z \in Z \ :\ v + z \ \geq \ b, \ \ v \geq 0 \}.$$

Lemma 1 *If $\hat{b} \neq 0$, then all points in Q^1 satisfy the* basic mixed-integer inequality

$$\frac{v}{\hat{b}} + z \geq \lceil b \rceil . \tag{1}$$

Proof: Let $(v, z) \in Q^1$.
Case 1: $z \geq \lceil b \rceil$. Then $v \geq 0 \Rightarrow \ v/\hat{b} + z \geq \lceil b \rceil$.
Case 2: $z \leq \lfloor b \rfloor$. Then $v + z \geq b \Rightarrow \ v \ \geq \ \hat{b} + \lfloor b \rfloor - z \ \geq \ \hat{b} + \hat{b}(\lfloor b \rfloor - z) \ = \ \hat{b}(\lceil b \rceil - z)$. ■

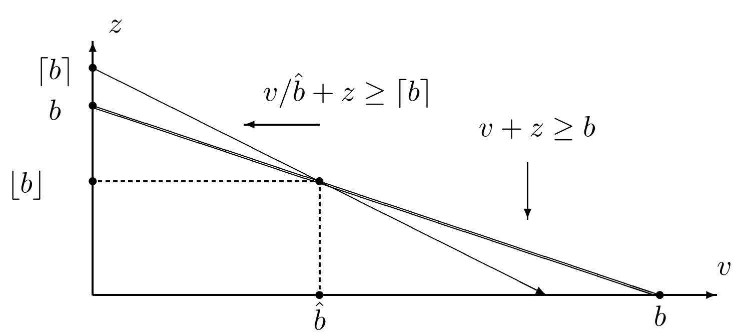

Figure 1: basic MIR inequality.

In Figure 1, the point $(\bar{v}, \bar{z}) = (0, b)$ violates the basic MIR inequality when b is not integral.

Lemma 2 *Let $\hat{b} \neq 0$. Then*

$$conv(Q^1) = \{ v, z \in R \ :\ v + z \ \geq \ b, \ \ \frac{v}{\hat{b}} + z \geq \lceil b \rceil , \ \ v \geq 0 \}.$$

Therefore the basic mixed-integer inequality is the only non-trivial facet defining inequality for Q^1. If $\hat{b} = 0$, then Q^1 has no non-trivial facet.

1.4 Applying the basic MI inequality

Let $S \subseteq I$. Starting with the equation defining Q, we can round up the coefficients of $x_j, j \in I \setminus S$, and drop continuous variables with negative coefficients to obtain

$$\sum_{c_j>0} c_j v_j + \sum_{j \in S} a_j \, x_j + \sum_{j \in I \setminus S} \lceil a_j \rceil \; x_j \geq b,$$

a valid inequality for Q. Writing $a_j = \lfloor a_j \rfloor + \hat{a}_j$ for $j \in S$, and re-arranging terms, we get

$$\Big(\sum_{c_j>0} c_j v_j + \sum_{j \in S} \hat{a}_j \, x_j \Big) + \Big(\sum_{j \in S} \lfloor a_j \rfloor \; x_j + \sum_{j \in I \setminus S} \lceil a_j \rceil \; x_j \Big) \geq b.$$

The first part of this inequality is non-negative, and the second part is integral for all $(v, x) \in Q$. Applying the basic mixed-integer inequality we get:

$$\frac{1}{\hat{b}} \Big(\sum_{c_j>0} c_j v_j + \sum_{j \in S} \hat{a}_j \, x_j \Big) + \Big(\sum_{j \in S} \lfloor a_j \rfloor \; x_j + \sum_{j \in I \setminus S} \lceil a_j \rceil \; x_j \Big) \geq \lceil b \rceil .$$

The coefficients of x_j in this inequality are:

$$\begin{array}{rll} \lfloor a_j \rfloor + \dfrac{\hat{a}}{\hat{b}} & \text{if} & j \in S, \\ \lceil a_j \rceil & \text{if} & j \in I \setminus S. \end{array}$$

The best choice for the set S is $S = \{j \in I : \hat{a}_j \leq \hat{b}\}$. Then the above inequality becomes

$$\frac{1}{\hat{b}} \sum_{c_j>0} c_j v_j + \sum_{j \in I} (\lfloor a_j \rfloor + \min\{\frac{\hat{a}_j}{\hat{b}}, 1\}) \, x_j \geq \lceil b \rceil .$$

We call this the mixed-integer rounding (MIR) inequality.

Definition 3 *The MIR function with parameter $b \neq 0$ is defined as:*

$$f^b(v) = \min\{\frac{\hat{v}}{\hat{b}}, 1\}.$$

1.5 Scaled MIR inequalities

We can scale the equation defining Q by a rational number t before writing the MIR inequality to obtain the t-scaled MIR inequality:

$$\frac{1}{\hat{b}^t} \sum_{tc_j>0} tc_j v_j + \sum_{j \in I} (\lfloor ta_j \rfloor + f^{tb}(ta_j)) \, x_j \geq \lceil tb \rceil . \tag{2}$$

Definition 4 *For a non-zero number t, the t-scaled MIR function with parameter b, is defined by $f^{t,b}(v) = f^{tb}(tv)$.*

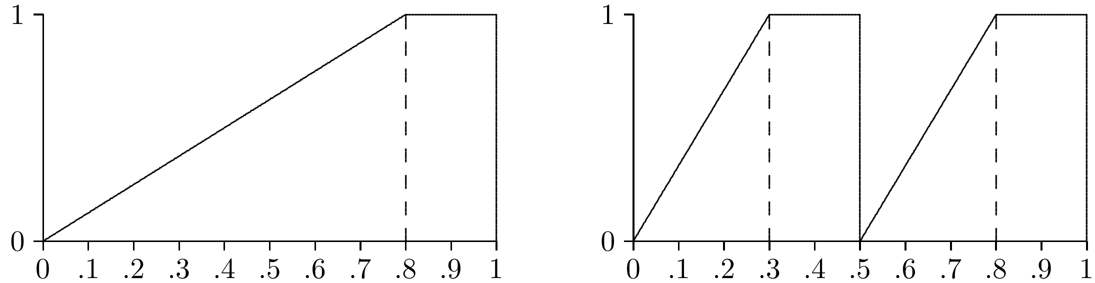

Figure 2: MIR functions $f^{0.8}(\cdot)$ and $f^{2,0.8}(\cdot)$.

Subtracting $t(\sum_{j\in J} c_j v_j + \sum_{j\in I} a_j\, x_j) = tb$ from the t-scaled MIR inequality, and dividing by $(1-\hat{b}^t)$, we get an equivalent inequality

$$\frac{1}{\hat{b}^t}\sum_{tc_j>0} tc_jv_j - \frac{1}{1-\hat{b}^t}\sum_{tc_j<0} tc_jv_j + \sum_{j\in\bar{S}} \frac{\hat{a}_j^t}{\hat{b}^t}\, x_j + \sum_{j\notin\bar{S}} \frac{1-\hat{a}_j^t}{1-\hat{b}^t}\, x_j \geq 1,$$

Lemma 5 *Assume the equation defining Q is rational, and let n be the smallest integer such that $nb \in Z$ and $na_j \in Z$ for all $j \in I$. Then, there are at most $\lfloor n/2 \rfloor$ distinct t-MIR cuts for Q, with integral t. In particular,*

a. If $t < 0$, then the t-MIR cut is the same as the $(-t)$-MIR cut, and,

b. if $t > n$, then the t-MIR cut is the same as the $(t-n)$-MIR cut, and,

c. if $n > t > \lfloor n/2 \rfloor$ then the t-MIR cut is the same as the $(n-t)$-MIR cut.

Proof Define $h^{t,b}(v) = \hat{v}^t/\hat{b}^t$ if $\hat{v}^t \leq \hat{b}^t$, and $(1-\hat{v}^t)/(1-\hat{b}^t)$ otherwise. It is easy to see that (i) $h^{t,b}(v) = h^{-t,b}(v)$ for all $v \in R$, implying (a), and (ii) $h^{t,b}(v) = h^{t-n,b}(v)$ whenever v is an integral multiple of $1/n$, implying (b). From (i) and (ii) it follows that $h^{t,b}(v) = h^{n-t,b}(v)$ for v an integral multiple of $1/n$, implying (c). Finally, $t = 0$ or $t = n$ does not lead to a valid inequality, because $\hat{b}^t = 0$, therefore the only values of t that can give distinct t-MIR cuts are $1, 2, \ldots, \lfloor n/2 \rfloor$. ∎

The above approach used in deriving scaled-MIR inequalities involves mapping

$$\mathcal{T} : Q \rightarrow Q' = \{\begin{pmatrix} v' \\ z' \end{pmatrix} = A\begin{pmatrix} v \\ x \end{pmatrix} + b\} \subseteq Q^1,$$

where A is a matrix, b is a vector. If $\alpha^T \begin{pmatrix} v' \\ z' \end{pmatrix} \geq \beta$ is valid for Q', then $\alpha^T A \begin{pmatrix} v \\ x \end{pmatrix} + \alpha^T b \geq \beta$ is valid for Q. This approach is used in *local cuts* for the TSP by Applegate et. al. [**?**] with varying A. Espinoza [6] also generates A dynamically to generate cuts for Q.

1.6 A simple polyhedral set

In this section we study simple mixed-integer sets with three variables. The subsequent results are from Dash and Günlük (2006). We first look at

$$Q^{2+} = \{v \in R,\ y,z \in Z \ :\ v + \alpha y + z \ \geq \ \beta, \quad v,y,z \geq 0\},$$

when $\alpha, \beta \in R$ and satisfy $1 > \beta > \alpha > 0$, and $\lceil \beta/\alpha \rceil > \beta/\alpha$. Note that variable z is required to be non-negative. Under our restrictions on α and β, it is possible to describe the convex hull of Q^{2+} explicitly. As we discuss later, the resulting inequalities are valid for the general case under mild conditions. First we show that the following MIR based inequalities are facet defining for Q^{2+}.

Lemma 6 *The following inequalities are valid and facet defining for Q^{2+}:*

$$v + \alpha y + \beta z \ \geq \ \beta, \tag{3}$$

$$\big(1/(\beta - \alpha \lfloor \beta/\alpha \rfloor)\big) v + y + \lceil \beta/\alpha \rceil z \ \geq \ \lceil \beta/\alpha \rceil . \tag{4}$$

Proof Note that inequality (3) can be obtained by treating $v+\alpha y$ as a continuous variable and writing the MIR inequality based on $(v + \alpha y) + z \ \geq \ \beta$. To obtain inequality (4), we start with inequality (3), divide it by α and relax the resulting inequality as follows:

$$v/\alpha + y + \lceil \beta/\alpha \rceil z \ \geq \ \beta/\alpha \tag{5}$$

Writing the MIR inequality where v/α is treated as a continuous variable and $y + \lceil \beta/\alpha \rceil z$ is treated as an integer variable gives inequality (4).

To see that the inequalities are facet defining, consider the following distinct points:

$$p_1 = (0,0,1), \quad p_2 = (0, \lceil \beta/\alpha \rceil, 0), \quad p_3 = (\beta - \alpha \lfloor \beta/\alpha \rfloor, \lfloor \beta/\alpha \rfloor, 0), \quad p_4 = (\beta, 0, 0)$$

and note that (i) $p_1, p_3, p_4 \in Q^{2+}$ are affinely independent and satisfy inequality (3) with equality, (ii) $p_1, p_2, p_3 \in Q^{2+}$ are affinely independent and satisfy inequality (4) with equality. ■

Inequality (3) and inequality (4) are sufficient to obtain the convex hull of Q^{2+}. We call inequality (4) *the two-step MIR inequality.*

Lemma 7 *$conv(Q^{2+}) = T$, where*

$$T \ = \ \{v,y,z \in R \ :\ v,y,z \ \textit{satisfy}\ (3),\ (4), \quad v,y,z \geq 0\}.$$

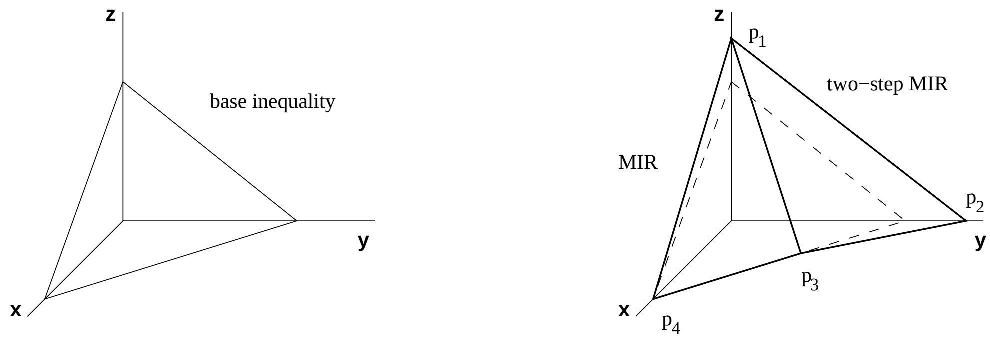

Figure 3: Facets of Q^{2+}

As shown in Figure 3, points p_1, p_2, p_3, p_4 are the only extreme points of $conv(Q^{2+})$, and the unit vectors give the only extreme directions of Q^{2+}.

Lemma 8 *If $1/\alpha \le \lceil \beta/\alpha \rceil$, inequality (4) is an $(1/\alpha)$-scaled MIR inequality for Q^{2+}.*

Let Q^2 be a relaxation of Q^{2+} obtained by allowing the z variable to assume negative values. More precisely,

$$Q^2 = \{v \in R,\ y, z \in Z \ : \ v + \alpha y + z \ \ge \ \beta, \quad v, y \ge 0\}.$$

and remember that $\beta, \alpha \in R$ satisfy $1 > \beta > \alpha > 0$, and $\lceil \beta/\alpha \rceil > \beta/\alpha$.

Even though β is required to be less than 1 in Q^2, the fact that z can take on negative values makes the set fairly general. Some of the complexity of the facial structure of $conv(Q^{2+})$ when β is large can be captured by studying the set Q^2. We next show that inequalities (3) and (4) are facet defining for Q^2 under mild conditions. They are not necessarily sufficient to describe the convex hull.

Lemma 9 *Inequality (3) is facet defining for Q^2. In addition, inequality (4) is facet defining for Q^2 if $1/\alpha \ge \lceil \beta/\alpha \rceil$.*

Proof As $p_1, p_2, p_3, p_4 \in Q^{2+} \subseteq Q^2$, we only need to show validity of inequality (3) and (4). Inequality (3) is valid as it is derived in the proof of Lemma 6 without assuming $z \ge 0$

To see that inequality (4) is valid, notice that the inequalities

$$\begin{aligned} (1/\alpha)v \ + \ y + \ \ (\beta/\alpha)z \ &\ge \ \beta/\alpha, \\ (1/\alpha)v \ + \ y + \ \ (1/\alpha)z \ &\ge \ \beta/\alpha, \end{aligned}$$

are valid for Q^2, and therefore, for any $\gamma \in R$ satisfying $1/\alpha \geq \gamma \geq \beta/\alpha$, the inequality

$$(1/\alpha)v \; + \; y + \; \; \gamma z \; \; \geq \; \beta/\alpha$$

is also valid since it can be obtained as a convex combination of valid inequalities. Since $1/\alpha \geq \lceil \beta/\alpha \rceil$ by assumption, 5 is valid for Q^2, and therefore so is inequality (4). ■

1.7 Two-step MIR inequalities for Y

We present an example to illustrate how these inequalities can be applied.

Example 10 *Consider the equation* $1.2x_1 + 3.35x_2 + 2.5x_3 + 0.8x_4 + x_5 = 4.7$ *with all variables non-negative and integral. This equals* $0.2x_1 + 0.35x_2 + 0.5x_3 + 0.8x_4 + w = 0.7$, *where* $w = x_1 + 3x_2 + 2x_3 + x_5 - 4$. *Here* $\hat{b} = 0.7$; *let* $\alpha = 0.4$. *Then* $1/\alpha = 2.5 > \lceil \hat{b}/\alpha \rceil = 2$, *and* $\rho = 0.3$. *Since* x_2 *and* x_4 *are non-negative, the inequality*

$$(0.2x_1 + 0.1x_3) + 0.4(x_2 + x_3) + (x_4 + w) \geq 0.7$$

is a relaxation of the previous equation. Of the three terms in brackets, the first two are non-negative, and the last two are integral; we can thus apply inequality (4) to obtain

$$\frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{2}{3}x_3 + x_4 + w \geq 1$$

as a cutting plane.

We next generalize this procedure. First assume all c_js in Q are zero. Define $\tau = \lceil \hat{b}/\alpha \rceil \geq 2$, and define $\rho = \hat{b} - \alpha \lfloor \hat{b}/\alpha \rfloor$. Let k_j, l_j be integers such that $k_j \leq \lfloor \hat{a}_j/\alpha \rfloor$, and $l_j \geq \lceil \hat{a}_j/\alpha \rceil$, for $j \in I$.

Let I_0, I_1 and I_2 be sets which form a partition of I. We can relax the equation in Q to obtain

$$\sum_{j \in I_1} (k_j \alpha + (\hat{a}_j - k_j \alpha))\, x_j + \sum_{j \in I_2} l_j \alpha\, x_j + \sum_{j \in I_0} x_j + w \quad \geq \quad \hat{b},$$

which can be rewritten as

$$\sum_{j \in I_1} (\hat{a}_j - k_j \alpha)\, x_j + \alpha \Big(\sum_{j \in I_1} k_j\, x_j + \sum_{j \in I_2} l_j\, x_j \Big) + \Big(\sum_{j \in I_0} x_j + w \Big) \quad \geq \quad \hat{b},$$

Applying inequality (4) and substituting for w leads to inequality

$$\sum_{j \in I} \gamma_j x_j + \rho \tau\, z \geq \rho \tau \lceil b \rceil, \tag{6}$$

where

$$\gamma_j = \rho\,\tau\,\lfloor a_j \rfloor + \begin{cases} \rho\,\tau & \text{if } j \in I_0 \\ k_j\rho \ + \hat{a}_j - k_j\alpha & \text{if } j \in I_1 \\ l_j\rho & \text{if } j \in I_2 \end{cases}$$

By inspection, the strongest inequality of this form is obtained by setting $k_j = k_j^* = \lfloor \hat{a}_j/\alpha \rfloor$ and $l_j = l_j^* = \lceil \hat{a}_j/\alpha \rceil$ for $j \in I$, and letting

$$I_0 = \{j \in I \ : \ \hat{a}_j \geq \hat{b}\},$$

and

$$I_1 = \{j \in I \setminus I_0 \ : \ \hat{a}_j - k_j^*\alpha < \rho\}, \qquad I_2 = \{j \in I \setminus I_0 \ : \ \hat{a}_j - k_j^*\alpha \geq \rho\}.$$

In other words,

$$\gamma_j = \rho\,\tau\,\lfloor a_j \rfloor + \min\{\rho\,\tau \ , \ k_j^*\rho \ + \hat{a}_j - k_j^*\alpha \ , \ l_j^*\rho \ \}.$$

If the continuous variables have non-zero coefficients, then the two-step MIR inequality for Q becomes

$$\sum_{c_j>0} c_j v_j + \sum_{j\in I} \gamma_j x_j + \rho\,\tau\,z \geq \rho\,\tau\,\lceil b \rceil, \tag{7}$$

These inequalities imply the inequalities of Letchford and Lodi (2002). They also equal some classes of inequalities for Q obtained by Atamtürk [2] via sequential lifting, if the variables x have no upper bounds.

Theorem 11 *(Dash, Günlük 2003) Given a point $(v^*, x^*) \in Q^{LP}$, the most violated two-step MIR inequality can be found in polynomial time, assuming $\tau \leq t$ for some fixed t.*

2 Lecture 2

We discuss how to obtain cutting planes for mixed-integer programs from Gomory's master cyclic group polyhedron .

2.1 Gomory's master cyclic group polyhedron

In the previous lecture, we discussed the set

$$Q \;=\; \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i\, x_i = b, \quad x, v \geq 0,\; x \text{ integer} \Big\}.$$

Rewrite Q as

$$Q \;=\; \Big\{ v \in R^{|J|}, x \in Z^{|I|} \;:\; (\sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor) + \sum_{j \in J} c_j v_j + \sum_{i \in I} \hat{a}_i\, x_i = \hat{b}, \quad x, v \geq 0 \Big\}$$

and assume that $\hat{a}_i$ $(i \in I)$ and $\hat{b}$ are rational numbers with a common denominator n, and let $\hat{b} = r/n$, where $0 < r < n$. We call n the *scaling factor* for Q.

Let $I_k = \{ i \in I \,:\, \hat{a}_i = k/n \}$ and define the mapping

$$\begin{aligned} w_k &= \sum_{i \in I_k} x_i, \\ v_+ &= \sum_{c_j \geq 0} c_j v_j, \quad v_- = \sum_{c_j < 0} c_j v_j, \\ z &= \sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor \end{aligned} \tag{8}$$

that maps each point (v, x) in Q to a point (v_+, v_-, w) in the polyhedron

$$P'(n, r) = conv\{ v_+, v_- \in R,\; w \in Z^{n-1} : v_+ - v_- + \sum_{i=1}^{n-1} \frac{i}{n}\, w_i - z = \frac{r}{n}, \;\; v_+, v_-, w \geq 0, z \in Z \}.$$

(If I_k is empty for some k, set w_k to zero.) If Q has no continuous variables, then (8) maps points in Q to points in the *master cyclic group polyhedron* of Gomory:

$$P(n, r) = conv\{\; w \in Z^{n-1} \;:\; \sum_{i=1}^{n-1} \frac{i}{n}\, w_i - z = \frac{r}{n}, \quad v_+, v_-, w \geq 0, z \in Z \}.$$

We view $P'(n, r)$ as the mixed-integer extension of $P(n, r)$. We call n the *size* of $P'(n, r)$. Thus the scaling factor of Q equals the size of the associated $P'(n, r)$ (or $P(n, r)$). The constraint defining $P(n, r)$ can be written as

$$\sum_{i=1}^{n-1} i w_i \equiv\; r \,(\bmod\, n).$$

Further, for $P(n, r)$ one can assume z is non-negative, but not for $P'(n, r)$.

2.2 Basic properties

Gomory [8] characterized the *polar* of $P(n,r)$, i.e., the convex hull of nontrivial facets (excluding the non-negativity inequalities) of $P(n,r)$. We assume all facet defining inequalities have a right hand side of 1.

Theorem 12 (Gomory [8]) *If $r \neq 0$, then $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ is a non-trivial facet of $P(n,r)$ if and only if $\eta = (\eta_j)$ is an extreme point of the inequalities*

$$\begin{aligned}
\eta_i + \eta_j &\geq \eta_{(i+j) \bmod n} \quad \forall i,j \in \{1,\ldots,n-1\}, &(9)\\
\eta_i + \eta_j &= \eta_r \quad \forall i,j \text{ such that } r = (i+j) \bmod n, &(10)\\
\eta_j &\geq 0 \quad \forall j \in \{1,\ldots,n-1\}, &(11)\\
\eta_r &= 1. &(12)
\end{aligned}$$

The property (9) is called *sub-additivity*; we call (10) *r-additivity.*

Theorem 13 *(Gomory, Johnson [9]) An inequality $\eta^T(v_+, v_-, w) \geq 1$ defines a facet of $P'(n,r)$ if and only if it has the form*

$$n\eta_1 v_+ + n\eta_{n-1} v_- + \sum_{i=1}^{n-1} \eta_i w_i \geq 1, \tag{13}$$

where $(\eta_1, \ldots, \eta_{n-1})$ defines a facet of $P(n,r)$.

Valid inequalities for $P'(n,r)$ yield valid inequalities for Q. Given a facet (13) of $P'(n,r)$,

$$n\eta_1(\sum_{c_j \geq 0} c_j v_j) + n\eta_{n-1}(\sum_{c_j<0} c_j v_j) + \sum_{i \in I} f(\hat{a}_i) x_i \geq 1, \tag{14}$$

is a valid inequality for Q, where $f(\hat{a}_i) = \eta_k$ if $\hat{a}_i = k/n$. We call such inequalities *group cuts* for Q. We will see that the GMI cut can be derived as a group cut.

Let $(v', x') \in Q^{LP} \setminus conv(Q)$, and let (v', x') be mapped to (v'_+, v'_-, w') via (8). The most violated group cut for (v', x'), if there is one, can be found by solving the *separation LP*

$$\min\left\{(nv'_+)\eta_1 + (nv'_-)\eta_{n-1} + \sum_i w'_i \eta_i \;:\; \eta \text{ satisfies } (9) - (12)\right\}.$$

If the optimum value of the separation LP is less than 1, then $(v'_+, v'_-, w') \notin P'(n,r)$ and the corresponding group cut (14) is violated by (v', x'). If the optimum value of the separation LP is at least 1, then $(v'_+, v'_-, w') \in P'(n,r)$ and (v', x') satisfies all group cuts.

We need some definitions before proving Theorems 12 and 13. For $P'(n,r)$, let e_i ($i = 1,\ldots,n-1$), e_+ and e_- be the unit vectors with ones in, respectively, the w_ith component, the v_+ component, and the v_- component, and zeros elsewhere. For a vector η in the $P(n,r)$ space, define its extension in the $P'(n,r)$ space as $\eta_{ex} = (n\eta_1, n\eta_{n-1}, \eta)$. For a vector η' in the $P'(n,r)$ space, define its restriction to the $P(n,r)$ space by removing the v_+ and v_- components, and denote it by η'_{re}.

Lemma 14 *Both $P(n,r)$ and $P'(n,r)$ are full-dimensional unbounded polyhedra, and their characteristic (recession) cones are R_+^{n-1} and R_+^{n+1} respectively.*

Lemma 14 implies that if $\eta^T w \geq 1$ is a facet defining inequality of $P(n,r)$, then $\eta \geq 0$. Further, any non-trivial facet has the following property: for $i = 1,\ldots,n-1$, there is a set of $n-1$ linearly independent integer points χ^i in $P(n,r)$ satisfying $\chi_i^i \geq 1$, for $i = 1,\ldots,n-1$.

Lemma 15 *Let $\eta \in R^{n-1}$ satisfy the constraints (10) - (12). If $\eta^T x \geq 1$ is a valid inequality for $P(n,r)$, then η satisfies the sub-additivity constraints (9).*

Proof Let $j,k \in \{1,,\ldots,n-1\}$, and let $i = j + k \bmod n$. Let $\chi = e_j + e_k + e_{r-i}$. Then

$$\begin{aligned} \chi \in P(n,r) \quad &\Rightarrow \quad \eta^T\chi \ = \ \eta_j + \eta_k + \eta_{r-i} \ \geq \ 1 \ = \ \eta_i + \eta_{r-i} \\ &\Rightarrow \quad \eta_j + \eta_k \geq \eta_i. \end{aligned}$$

∎

Proof of Theorem 12 Let $\gamma^T w \geq 1$ define a non-trivial facet of $P(n,r)$. We argued earlier that $\gamma \geq 0$. We next show that γ satisfies (10) and (12). Observe that

$$e_r,\ e_i + e_{r-i} \in P(n,r) \Rightarrow \gamma_r \geq 1,\ \gamma_i + \gamma_{r-i} \geq 1.$$

Further, there are integral points χ', χ'' and χ''' in $P(n,r)$ lying on this facet such that $\chi'_i \geq 1$ and $\chi''_{r-i} \geq 1$ and $\chi'''_r \geq 1$ Then $\gamma^T(\chi' + \chi'') = 2$. But

$$\chi = \chi' + \chi'' - e_i - e_{r-i} \in P(n,r) \Rightarrow \gamma^T\chi = 2 - \gamma_i - \gamma_{r-i} \geq 1 \Rightarrow \gamma_i + \gamma_{r-i} \leq 1.$$

Similarly $\gamma^T\chi''' = 1$, and

$$\bar{\chi} = 2\chi''' - e_r \in P(n,r) \Rightarrow \gamma^T\bar{\chi} = 2 - \gamma_r \geq 1 \Rightarrow \gamma_r \leq 1.$$

Therefore γ satisfies (10) - (12). By Lemma 15, it also satisfies the sub-additivity constraints.

Let η satisfy (9) - (12). For any point integral $w^* \in P(n,r)$

$$\sum_i \eta_i w_i^* \geq \sum_i \eta_{(iw_i^*)} \geq \eta_{(\sum_i iw_i^*)} = \eta_r = 1,$$

where the subscripts inside the brackets are computed modulo n. Therefore $\eta^T w \geq 1$ defines a valid inequality for $P(n,r)$. Let

$$P(n,r) = \{w \in R^{n-1} : Aw \geq \mathbf{1},\ w \geq 0\},$$

where $Aw \geq 1$ represents the non-trivial facets of $P(n,r)$. This implies that if A_j stands for the jth column of A, then for any index $i \neq r$, $A_i + A_{r-i} = A_r = \mathbf{1}$. Observe that

$$\min\{\eta^T w : Aw \geq \mathbf{1},\ w \geq 0\} = 1,$$

as $e_r \in P(n,r)$ and $\eta^T e_r = \eta_r = 1$. Therefore, an optimal dual vector y of the above LP satisfies

$$y^T A \leq \eta, \quad y^T \mathbf{1} = 1, \quad y \geq 0.$$

If $y^T A_i < \eta_i$ for any index $i \neq r$, then

$$1 = \eta_i + \eta_{r-i} > y^T A_i + y^T A_{r-i} = y^T \mathbf{1} = 1.$$

Therefore $y^T A_i = \eta_i$ for $i = 1, \dots, n-1$. This implies that η is a convex combination of non-trivial facets $\{\eta_i\}$ of $P(n,r)$, which in turn are solutions of (9) - (12). Thus, if η is an extreme point (9) - (12), η is a facet of $P(n,r)$, and the vector y is a unit vector. If a facet is not an extreme point of (9) - (12), it is a convex combination of solutions of this system, and therefore of valid inequalities for $P(n,r)$. ∎

Lemma 16 *If $\chi = (v'_+, v'_-, w')$ is a vertex of $P'(n,r)$, then (i) both v'_+ and v'_- cannot be non-zero, and (ii) v'_+ and v'_- are integral multiples of $1/n$.*

Proof (i) If v'_+ and v'_- are non-zero, and $\min\{v'_+, v'_-\} = \epsilon$, then

$$\chi_1 = \chi + \epsilon(1,1,0,\dots,0) \text{ and } \chi_2 = \chi - \epsilon(1,1,0,\dots,0)$$

belong to $P'(n,r)$ and $\chi = (\chi_1 + \chi_2)/2$, which implies that χ is not a vertex.
(ii) If v'_- is non-zero, then $v'_+ + \sum_i w'_i/n - z = r/n$ for some integer z which implies that v'_+ is a multiple of $1/n$. A similar argument holds for v'_-. ∎

Proof of Theorem 13 Let $\eta^T x \geq 1$ be a non-trivial facet of $P(n,r)$. Let $\chi = (v'_+, v'_-, w')$ be a vertex of $P'(n,r)$, and assume $v'_+ > 0$. Then nv'_+ is an integer and $v'_- = 0$. The vector η_{ex} defines an inequality (with r.h.s. 1) satisfied by χ as

$$\eta_{ex}^T \chi = \eta_1 n v'_+ + \sum_i \eta_i w'_i \ \geq\ \eta_{(nv'_+)} + \eta_{(\sum_i iw'_i)} \ \geq\ \eta_{(nv'_+ + \sum_i iw'_i)} = \eta_r = 1.$$

The subscripts inside the brackets above are taken modulo n. The inequalities above follow from sub-additivity. Further, $\eta_{ex} \geq 0$ and the recession cone of $P'(n,r)$ equals R_+^{n+1} and therefore $\eta_{ex}^T(v_+, v_-, w) \geq 1$ is a valid inequaliy for $P'(n,r)$. To see that this is a facet of $P'(n,r)$, let $\chi^1, \ldots \chi^{n-1}$ be linearly independent points in $P(n,r)$ which satisfy $\eta^T\chi^i = 1$. There are indices j, k such that $\chi_1^j = s > 0$ and $\chi_{n-1}^k = t > 0$. Define $n+1$ linearly independent points in R^{n+1} as follows:

$$\psi^i = (0, 0, \chi^i) \text{ for } i = 1, \ldots, n-1;$$

$$\psi^+ = \psi^j + \frac{s}{n}e_+ - se_1;\ \psi^- = \psi^k + \frac{t}{n}e_- - te_{n-1}.$$

These points satisfy $\eta_{ex}^T\chi \geq 1$ with equality, and therefore η_{ex} defines a facet of $P'(n,r)$.

Assume η defines a facet of $P'(n,r)$. There exists a vertex $\chi \in P'(n,r)$ incident with this facet (i.e., $\eta^T\chi = 1$) with a positive v_+ component. Then $\chi_{v+} = t/n$ for some positive integer t. Let χ' in $P'(n,r)$ be obtained from χ by setting χ'_{v+} to 0 and χ'_1 to t. Then $\eta^T\chi' \geq 1 \Rightarrow n\eta_{v_+} \leq n\eta_1$. By modifying a vertex χ of $P'(n,r)$ with $\chi_1 > 0$, we can similarly conclude that $\eta_{v_+} \geq n\eta_1$ and therefore $\eta_{v_+} = n\eta_1$. The argument for $\eta_{v_-} = n\eta_{n-1}$ is identical. Finally, any point in $P(n,r)$ can be mapped to a point in $P'(n,r)$ by appending zeros in the v_+ and v_- components, and therefore $\eta_{re}^T w \geq 1$ is a valid inequality for $P(n,r)$. If it is not a facet, then $\eta_{re} \geq \sum_i \lambda_i\eta^i$ for some facets η^i of $P(n,r)$, and some numbers $\lambda_i \geq 0$. But that would imply that $\eta \geq \lambda_i\eta_{ex}^i$. By the first part of the proof, the vectors η_{ex}^i define facets of $P'(n,r)$, and this contradicts the assumption that η defines a facet of $P'(n,r)$. ∎

Remark 17 *Gomory and Johnson [9] showed that $P'(n,r)$ has exponentially many facets (in n). For example, when $n = 50$ and $r = 10$, $P'(n,r)$ has at least 74,000 facets, see [5].*

2.3 Facets of $P(n,r)$

As the defining equation for $P(n,r)$ has the same form as Q^1, we can define valid inequalities for $P(n,r)$ by applying the MIR cut to $P(n,r)$.

Definition 18 *The t-scaled (two-step) MIR inequality for $P(n,r)$ for an integer $t > 0$ is the inequality obtained after applying the t-scaled (two-step) MIR inequality to*

$$Q_p = \{w_i, z \in Z : \sum_{i=1}^{n-1} \frac{i}{n}w_i - z = \frac{r}{n}, w \geq 0, z \geq 0\}$$

and then substituting out z.

Therefore one can obtain t-scaled MIR inequalities and t-scaled two-step MIR inequalities as

$$\sum_{i=1}^{n-1}(\lfloor ti/n\rfloor + f^{tr/n}(ti/n))\, w_i \geq 1, \tag{15}$$

$$\sum_{i=1}^{n-1}(\lfloor ti/n\rfloor + g^{tr/n,\Delta/n}(ti/n))\, w_i \geq 1 \tag{16}$$

for rational t and Δ.

For a positive integer n and integers t and i, define $(ti)_n = ti \bmod n$ (here, $k \bmod n$ stands for $k - n\lfloor k/n\rfloor$). The (1-scaled) MIR inequality for $P(n,r)$ is

$$\sum_{i\leq r}\frac{i}{r}w_i + \sum_{i>r}\frac{n-i}{n-r}w_i \geq 1, \tag{17}$$

and for an integer $t > 0$, the t-scaled MIR inequality (2) is

$$\sum_{(ti)_n<(tr)_n}\frac{(ti)_n}{(tr)_n}x_i + \sum_{(ti)_n\geq(tr)_n}\frac{n-(ti)_n}{n-(tr)_n}x_i \geq 1. \tag{18}$$

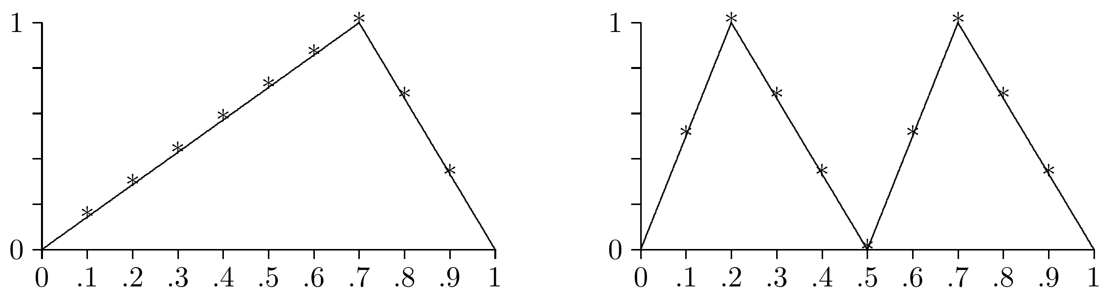

Figure 4: (i) The MIR facet for $P(10,7)$; (ii) The 2-scaled MIR facet for $P(10,7)$

Definition 19 *An inequality* $\sum_{i=1}^{n-1}\eta_i w_i \geq 1$ *is a* two-slope *inequality if* $\eta_i - \eta_{i-1}$ *equals either* η_1 *or* $-\eta_{n-1}$ *for* $i = 2,\ldots,n-1$.

Theorem 20 *(Gomory, Johnson [9]) Every two-slope inequality for* $P(n,r)$ *which satisfies the constraints (9) - (12) defines a facet for* $P(n,r)$.

Proof By definition, for $i = 2,\ldots,n-1$ either $\eta_1 + \eta_{i-1} = \eta_i$ or $\eta_{n-1} + \eta_i = \eta_{i-1}$. The constraints above are just the sub-additivity constraints (9) (satisfied as equalities) for $i \neq r, r+1$, and the r-additivity constraints (10) otherwise. These, along with the constraint $\eta_r = 1$ define $n-1$ linearly independent constraints from (9) - (12). ∎

Corollary 21 *(Gomory [8]) The MIR inequality for* $P(n,r)$ *defines a facet of* $P(n,r)$.

Proof Let $\eta^T w \geq 1$ stand for the MIR inequality. It clearly is a two-slope inequality, and satisfies (10) - (12). As it is a valid inequality for $P(n, r)$, Lemma 15 implies that it satisfies the sub-additivity constraints, and therefore the requirements of Theorem 20. ∎

Corollary 22 *The If $t > 0$ is a divisor of n and tr is not a multiple of n, then the t-scaled MIR inequality defines a facet of $P(n, r)$.*

Proof (Sketch) It is clear that the inequalities above are valid for $P(n, r)$. They are also two-slope inequalities and satsify (10) - (12). ∎

We show later that for every non-zero integer t, such that tr is not a multiple of n, the t-scaled MIR inequality (18) defines a facet of $P(n, r)$.

An automorphism ϕ is a bijection from $\{0, 1, \ldots, n-1\}$ to itself such that

$$\phi((a + b) \bmod n) = (\phi(a) + \phi(b)) \bmod n.$$

A bijection ϕ is an automorphism if and only if $\phi(i) = (ti)_n$ where t is coprime with n (n and t have no common divisors). The inverse of an automorphism is also an automorphism; if $\phi(i) = (ti)_n$, then $\phi^{-1}(i) = (ui)_n$ where u satisfies $tu \equiv 1 \pmod{n}$ (such a u exists as t and n are coprime).

Theorem 23 *(Gomory [8]) Let r be an integer such that $0 < r < n$. Let ϕ be an automorphism defined by $\phi(i) = (ti)_n$, and let $s = \phi(r)$. If $\sum_i \eta_i w_i \geq 1$ is a non-trivial facet of $P(n, r)$, then $\sum_i \eta_i w_{\phi(i)} \geq 1$ is a non-trivial facet of $P(n, s)$. Equivalently, $\sum_i \eta_{\phi^{-1}(i)} w_i \geq 1$ is a non-trivial facet of $P(n, s)$.*

Theorem 23 says that for an automorphism ϕ, the facets of $P(n, r)$ correspond to facets of $P(n, \phi(r))$ via a permutation of facet coefficients.

Example 24 *Let $w^* = (w_1^*, w_2^*, w_3^*, w_4^*)$ be non-negative integer vector satisfying*

$$1w_1 + 2w_2 + 3w_3 + 4w_4 \equiv 3 \bmod 5.$$

Note that 2 is co-prime with 5, and $3 = 2^{-1}$ in modular arithmetic, as $2 \times 3 \equiv 1 \bmod 5$. Clearly w^* also satisfies

$$2(1w_1 + 2w_2 + 3w_3 + 4w_4) \equiv 2 \times 3 \equiv 1 \bmod 5, \textit{ or}$$

$$2w_1 + 4w_2 + 1w_3 + 3w_4 \equiv 1 \bmod 5.$$

Also, any solution w' of the last expression satisfies

$$3(2w_1 + 4w_2 + 1w_3 + 3w_4) \equiv 3 \bmod 5,$$

which is the same as the first modular equation.

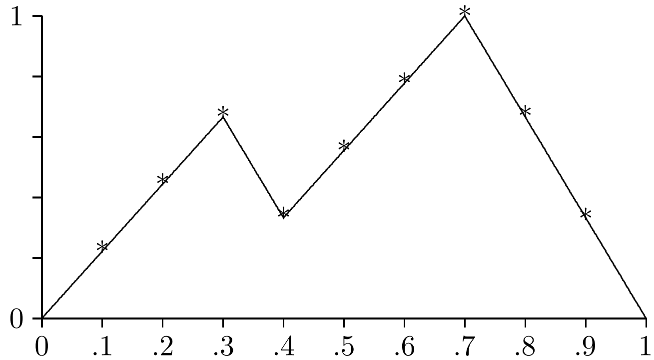

Figure 5: Two-slope facet for $P(10,7)$

2.4 Two-step MIR facets

When applied to $P(n,r)$, the two-step MIR inequalities yield many facets.

Theorem 25 *Let $\Delta \in Z^+$ be such that $r > \Delta > 0$. The two-step MIR inequality*

$$\sum_{i=1}^{n-1} g^{r/n,\Delta/n}(i/n)\, x_i \geq 1$$

defines a facet of $P(n,r)$ provided that $n > \Delta \lceil r/\Delta \rceil > r$. We call the corresponding facet the 1-scaled two-step MIR facet of $P(n,r)$ with parameter Δ.

Proof (Sketch) For $i \in I = \{1, 2, \ldots, n-1\}$ let $\eta_i = g^{r/n,\Delta/n}(i/n)$ so that the two-step MIR inequality can be written as $\sum_{i=1}^{n-1} \eta_i x_i \geq 1$.

First note that $\alpha = \Delta/n$ satisfies the conditions of Lemma 9: (i) $r > \Delta > 0 \Rightarrow \hat{b} = r/n > \alpha > 0$, and (ii) $n > \Delta \lceil r/\Delta \rceil > r \Rightarrow n/\Delta > \lceil r/\Delta \rceil > r/\Delta \Rightarrow 1/\alpha > \lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$. Therefore, the inequality is valid for $P(n,r)$. The rest of the proof is identical to the proof of Corollary 21. ∎

References

[1] A. Agra and M. F. Constantino, Description of 2-integer continuous knapsack polyhedra, to appear in *Discrete Optimization* (2006).

[2] A. Atamtürk, On the Facets of the Mixed-Integer Knapsack Polyhedron, *Mathematical Programming* **98** 145–175 (2003).

[3] A. Atamtürk and D. Rajan, Valid Inequalities for Mixed-Integer Knapsack from Two-Integer Variable Restrictions, IEOR, University of California–Berkeley, Technical report BCOL.04.02, (2004).

[4] S. Dash and O. Günlük, Valid inequalities based on simple mixed-integer sets, *Mathematical Programming*, **105**, 29–53, 2006.

[5] S. Dash and O. Günlük, Valid inequalities based on the interpolation procedure, *Mathematical Programming*, **106**, 111–136, 2006.

[6] D. Espinoza, Ph.D. Thesis, Industrial and Systems Engineering, Georgia Institute of Technology, 2006.

[7] S. Dash, O. Günlük and M. Goycoolea. Two step MIR inequalities for mixed-integer programs. *Manuscript*, 2005.

[8] R.E. Gomory, Some polyhedra related to combinatorial problems, Journal of Linear Algebra and its Applications, **2**, 451–558 (1969).

[9] R.E. Gomory and E. Johnson, Some continuous functions related to corner polyhedra I, *Mathematical Programming* **3** 23–85 (1972).

[10] R.E. Gomory and E. Johnson, Some continuous functions related to corner polyhedra II, *Mathematical Programming* **3** 359–389 (1972).

[11] A. N. Letchford and A. Lodi, Strengthening Chvátal-Gomory cuts and Gomory fractional cuts, *Operations Research Letters* **30(2)** 74–82 (2002).

[12] H. Marchand and L. Wolsey, Aggregation and mixed integer rounding to solve MIPs, *Oper. Res.* **49**, 363–371 (2001).

[13] L.A. Wolsey, *Integer Programming*, Wiley, New York (1998).