EXPONENTIAL SUMS WITH MULTIPLICATIVE COEFFICIENTS AND APPLICATIONS

RÉGIS DE LA BRETÈCHE AND ANDREW GRANVILLE

Abstract. We show that if an exponential sum with multiplicative coefficients is large then the associated multiplicative function is "pretentious". This leads to applications in the circle method, and a natural interpretation of the local-global principle.

1. Introduction

Diverse investigations in analytic number theory involve sums like

$$R_f(\alpha, x) := \sum_{n\leqslant x} f(n)\mathrm{e}(n\alpha)$$

where $\mathrm{e}(t) = \mathrm{e}^{2i\pi t}$ and $f(.)$ is a multiplicative function. For simplicity we will restrict our attention throughout to the class $\mathcal{M}$ of completely multiplicative functions f for which $|f(n)| \leqslant 1$ for all n; and let

$$F(s) = \sum_{n\geqslant 1} \frac{f(n)}{n^s}.$$

One can approximate any $\alpha \in \mathbb{R}/\mathbb{Z}$ by a rational $\frac{a}{q}$ with $(a, q) = 1$ and $q \leqslant Q$ (with, say, $Q = x/(\log x)^{2+\epsilon}$), so that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}.$$

If $q > (\log x)^{2+\epsilon}$ then α is on a *minor arc* and Montgomery and Vaughan [22] proved that

$$\sum_{n\leqslant x} f(n)\mathrm{e}(n\alpha) \ll \frac{x}{\log x}. \tag{1.1}$$

There are many examples of f that attain this bound.[1] In general we have the folklore conjecture

$$\sum_{n\leqslant x} f(n)\mathrm{e}(n\alpha) \ll \frac{x}{\log x} + \frac{x}{\sqrt{q}}. \tag{1.2}$$

The $\frac{x}{\sqrt{q}}$-term cannot, in general, be removed since there are many examples for which $R_f(\alpha, x) \gg \frac{x}{\sqrt{q}}$ when q is small. Our main goal in this paper is to classify these examples and to determine asymptotic formulae for $R_f(\alpha, x)$ in such cases. We will express $R_f(\alpha, x)$ in terms of other quantities that arise naturally in multiplicative number theory:

A.G. est partiellement soutenu par une bourse de la Conseil de recherches en sciences naturelles et en génie du Canada, and is also partially supported a European Research Council grant, agreement n° 670239. Part of this work was undertaken while A.G. was in Paris, his stay was supported by Université Paris–Diderot, Université Pierre et Marie Curie, Université d'Orsay and Fondation Sciences Mathématiques de Paris. The hospitality and financial support of these institutions is gratefully acknowledged. We thank Jörg Brüdern, K. Soundararajan and Trevor Wooley for useful discussions and correspondence.

[1]For example, no matter what $f(p)$ equals on the primes $p \leqslant \frac{x}{2}$ we select an angle θ and $f(p) = e(\theta - p\alpha)$ on the primes p, $\frac{x}{2} < p \leqslant x$, so that $e(\theta)$ points in the same direction as the sum of $f(n)$ over integers $n \leqslant x$ free of prime factors $> \frac{x}{2}$.

Theorem 1. *Let $f \in \mathcal{M}$ and $\alpha = a/q + \beta$ where $(a,q) = 1$ with $q \leqslant (\log x)^{2+\epsilon}$. There exists a primitive Dirichlet character $\chi \pmod r$ where r divides q, and a real number t with $|t| < \log x$ for which*

$$\sum_{n\leqslant x} f(n)\mathrm{e}(n\alpha) = \frac{\overline{\chi}(a)\kappa(q)g(\chi)}{\phi(q)} I(x,\beta,t) \sum_{n\leqslant x} \frac{f(n)\overline{\chi}(n)}{n^{it}} + O\Big(\frac{(1+|\beta|x)x}{(\log x)^{1-\frac{1}{\sqrt{2}}-o(1)}}\Big),$$

where $g(\chi)$ is the Gauss sum, κ is defined by the convolution $f(n)/n^{it} = (\kappa * \chi)(n)$, and we take

$$I(x,\beta,t) := \frac{1}{x}\int_0^x \mathrm{e}(\beta v) v^{it}\mathrm{d}v.$$

The character χ and the real number t are selected to maximize the sum on the right-hand side. If this sum remains larger than the error term for x in some range (like from X to X^2) then there is a unique possibility for χ which does not change as x varies, and t varies continuously if at all.

If $r = q$ with $f(n) = \chi(n)n^{it}$ when $(n,q) = 1$ then the main term here is

$$\frac{\overline{\chi}(a)f(q)q^{-it}\sqrt{q}}{\phi(q)} I(x,\beta,t) \sum_{\substack{n\leqslant x\\(n,q)=1}} 1 \ \sim\ \overline{\chi}(a)f(q)q^{-it}\cdot I(x,\beta,t)\cdot\frac{x}{\sqrt{q}}.$$

Since (trivially) $|\overline{\chi}(a)f(q)q^{-it}|,\ |I(x,\beta,t)| \leqslant 1$, this supports the folklore conjecture (1.2). The improved bound

$$|I(x,\beta,t)| \ll \frac{1}{\sqrt{1+|\beta|x}}$$

proved in (3.1), suggests our refined conjecture,

$$R_f(\alpha,x) \ll \frac{x}{\log x} + \frac{x}{\sqrt{q(1+|\beta|x)}}. \tag{1.3}$$

In Theorem 4 we will prove rather more than Theorem 1, obtaining an asymptotic series (of similar looking terms) with a better error term.

In the proof of Theorem 1 we write each $\mathrm{e}(n\alpha) = \mathrm{e}(\frac{an}{q})\mathrm{e}(n\beta)$ and replace the $\mathrm{e}(\frac{an}{q})$ by a sum over characters mod q; the same idea works for any bounded function of period q. Thus, for example, we also prove that if ξ is a character modulo squarefree q and $(abc,q) = 1$ then there exists a constant c_q, which depends on f,ξ,a,b,c with $|c_q| \ll e^{O(\omega(r))}\frac{q^2}{\phi(q)^2}$, such that

$$\sum_{\substack{n\leqslant x\\(n,q)=1}} f(n)\xi(n+c)\mathrm{e}\left(\frac{an+b\overline{n}}{q}\right) = \frac{c_q}{\sqrt{r}}\frac{x^{it}}{1+it}\sum_{n\leqslant x}\frac{f(n)\overline{\chi}(n)}{n^{it}} + O\Big(\frac{x}{(\log x)^{1-\frac{1}{\sqrt{2}}-o(1)}}\Big) \tag{1.4}$$

with χ and t as in Theorem 1. With additional care one can prove a version of this result in a range like $q \leqslant x^{1/2}$. We prove several more general and precise results in section 5.

Obtaining good estimates for $R_f(\alpha,x)$ on the major arcs allows us to obtain asymptotics in various Diophantine problems (weighted by multiplicative functions) using the circle method. The main term in these asymptotics typically involve an Euler product that can be decomposed into contributions from each prime (a *local-global principle*).

In the questions here the roles of small and large prime factors in the asymptiotic formulae are quite different and so we begin by splitting any $f \in \mathcal{M}$ into two multiplicative functions $f = F_sF_\ell$ where F_s involves only the "small" prime factors, and F_ℓ only the

"large", and we define

$$F_s(p) = \begin{cases} f(p) & \text{for } p \leqslant z, \\ \chi(p)p^{it} & \text{for } p > z, \end{cases}$$

where χ and t are defined as above, and we will take $z \asymp \log x$. Throughout we let $\eta = 1 - \frac{2}{\pi}$ for all $f \in \mathcal{M}$, and we can take $\eta = 1 - \frac{1}{\pi}$ if f is non-negative real-valued.

Corollary 1 (Corollary to Theorem 6)**.** *Suppose that A and B are sets of positive integers for which $A + B \subset \{1, 2, \dots, x\}$ with $|A||B| \geqslant x^2/(\log x)^{2\eta/3-\epsilon}$. Then*

$$\frac{1}{|A||B|} \sum_{\substack{a\in A \\ b\in B}} f(a+b) = \frac{1}{x} \sum_{n\leqslant x} F_\ell(n) \cdot \frac{1}{|A||B|} \sum_{\substack{a\in A \\ b\in B}} F_s(a+b) + o(1).$$

The mean value of multiplicative functions like $F_\ell(n)$ over $n \leqslant x$ is well-explored in the literature. The mean value of $F_s(a+b)$ over $a \in A, b \in B$, can be given explicitly (see Lemma 3 below) since it only varies unpredictably on the very small primes.

The circle method has typically been used to count solutions to equations in enough variables. Here we give solutions in two well-known problems in three variables:

Theorem 2. *Let $f, g, h \in \mathcal{M}$. Given positive integers a, b, c, we have*

$$\frac{1}{x^2/2} \sum_{\substack{\ell,m,n\leqslant x \\ a\ell+bm=cn}} f(\ell)g(m)h(n) = \frac{1}{x}\sum_{n\leqslant x} F_\ell(n) \cdot \frac{1}{x}\sum_{n\leqslant x} G_\ell(n) \cdot \frac{1}{x}\sum_{n\leqslant x} H_\ell(n)$$
$$\times \frac{1}{x^2/2} \sum_{\substack{\ell,m,n\leqslant x \\ a\ell+bm=cn}} F_s(\ell)G_s(m)H_s(n) + O\left(\frac{1}{(\log x)^{\tau/2-o(1)}}\right).$$

The main term is $o(1)$ unless $\chi_f = \chi_g = \chi_h = 1$.

Theorem 3. *Let $f, g, h \in \mathcal{M}$ and let $x = N \in \mathbb{N}$. Then*

$$\frac{1}{N^2/2} \sum_{\substack{\ell,m,n\geqslant 1 \\ \ell+m+n=N}} f(\ell)g(m)h(n) = \frac{1}{N}\sum_{n\leqslant N} F_\ell(n) \cdot \frac{1}{N}\sum_{n\leqslant N} G_\ell(n) \cdot \frac{1}{N}\sum_{n\leqslant N} H_\ell(n)$$
$$\times \frac{1}{N^2/2} \sum_{\substack{\ell,m,n\geqslant 1 \\ \ell+m+n=N}} F_s(\ell)G_s(m)H_s(n) + O\left(\frac{1}{(\log x)^{\tau/2-o(1)}}\right).$$

The main term is $o(1)$ unless $\chi_f = \chi_g = \chi_h = 1$.

The mean values of $F_s(\ell)G_s(m)H_s(n)$ over solutions to $a\ell+bm = cn$ or to $\ell+m+n = N$, can also be estimated by elementary methods as we will see in section 8.1. For example, suppose that A, B and C are the positive integers generated by given sets of primes with characteristic functions f, g and h, respectively. If $\delta_A := (1/x)\#\{a \leqslant x : a \in A\}$, and similarly δ_B and δ_C then Theorems 2 and 3 imply that

$$\begin{aligned} &\frac{1}{x^2/2}\#\{(\ell, m, n) \in A \times B \times C : \ell + m = n \leqslant x\} \\ &\qquad = \delta_A\delta_B\delta_C \prod_{p\notin A\cup B\cup C} \left(1 - \frac{1}{(p-1)^2}\right) + o(1), \end{aligned} \tag{1.5}$$

and

$$\frac{1}{N^2/2}\#\{(\ell,m,n)\in A\times B\times C\,:\,\ell+m+n=N\}$$
$$=\delta_A\delta_B\delta_C\prod_{\substack{p\notin A\cup B\cup C\\ p\nmid N}}\Big(1+\frac{1}{(p-1)^3}\Big)\prod_{\substack{p\notin A\cup B\cup C\\ p\mid N}}\Big(1-\frac{1}{(p-1)^2}\Big)+o(1). \tag{1.6}$$

It was shown in [12] that if f is a totally multiplicative function that only takes values 1 and -1 then there are at least $\frac12(1-\delta_0+o(1))x$ solutions to $f(n)=1$ with $n\leqslant x$ (and so no more than $\frac12(1+\delta_0+o(1))x$ solutions to $f(n)=-1$), and that this is best possible, where

$$\delta_0=-1+2\log(1+\sqrt{\mathrm{e}})-4\int_1^{\sqrt{\mathrm{e}}}\frac{\log t}{t+1}\mathrm{d}t=0.656999\ldots.$$

What about $f(a),f(b),f(c)$ for solutions to $a+b=c$? We apply Theorem 2 to prove

Corollary 2. *If $f,g,h\in\mathcal{M}$, taking only the values 1 and -1, then when x tends to ∞,*

$$\#\{1\leqslant a,b,c\leqslant x:\ a+b=c \text{ and } f(a)=g(b)=h(c)=-1\}\lesssim\kappa\cdot\tfrac12x^2$$

where $\kappa=\frac18(1+\delta_0)^3=.56869\ldots$, and

$$\#\{1\leqslant a,b,c\leqslant x:\ a+b=c \text{ and } f(a)=g(b)=h(c)=1\}\gtrsim\kappa'\cdot\tfrac12x^2$$

where $\kappa'=\frac18(1-\delta_0)^3=.005044\ldots$.

We use this result to bound the number of Pythagorean triples mod p up to any given point:[2] A proportion of at least κ' of the triples of residues $a,b,c\pmod p$ with $1\leqslant a,b,c\leqslant x<p$ and $a+b\equiv c\pmod p$, are c all quadratic residues mod p; moreover this proportion can be attained for some primes p, no matter how large x is.

The organization of this paper is a little complicated as there are lots of strands to bring together. In section 2 we will discuss what is known about mean values of multiplicative functions that is relevant to this paper and state the more general Theorem 4 in this context, from which Theorem 1 is deduced.

1.1. **More on $R_f(\alpha,x)$.** The best general bound for $R_f(\alpha,x)$ in the literature was given by Bachmann ([2], Theorem 3): If $|\alpha-\frac{a}{q}|\leqslant\frac{1}{q^2}$ with $(a,q)=1$ then

$$R_f(\alpha,x)\ll\frac{x}{\log x}+x\left(\frac{\log R\log\log R}{R}\right)^{1/2} \tag{1.7}$$

where $R=\min\{q,x/q\}$. One can easily deduce (1.1) (see the proof of Proposition 3 in section 10) except, perhaps, if $q\leqslant Q_1$ where

$$Q_1:=(\log x)^2(\log\log x)^{1+\varepsilon}\text{ and }\left|\alpha-\frac{a}{q}\right|\leqslant\frac{Q_1}{qx} \tag{1.8}$$

In fact La Bretèche ([7], Proposition 1) showed that if $R_f(\alpha,x)\gg x/\log x$ and f is not close to any $\psi(n)n^{it}$, then the $f(p)$ are suitably correlated for enough primes $p\gg x/\log x$ (see section 10.2 for further details).

We establish in section 10 that if (1.2) does not hold then

$$(\log x)^{2-o(1)}\leqslant q\leqslant(\log x)^2(\log\log x)^{1+o(1)},\quad\left|\alpha-\frac{a}{q}\right|\leqslant\frac{\log q\log\log q}{x}. \tag{1.9}$$

[2]Many thanks to Ben Green for suggesting this problem.

2. Known results on multiplicative functions

Let $t_f(x,T)$ denote a value of t which yields the maximum of

$$\left|F\left(1+\frac{1}{\log x}+it\right)\right| \tag{2.1}$$

as t runs through real numbers with $|t|\leqslant T$. Halász's Theorem (see [16] and [17]) gives upper bounds for $|\sum_{n\leqslant x} f(n)|$ in terms of the maximum of (2.1) where $t=t_f(x,\log x)$. In [17] it is observed that if $t=t_f(x,T)$ then

$$\sum_{n\leqslant x} f(n) = \frac{x^{it}}{1+it}\sum_{n\leqslant x}\frac{f(n)}{n^{it}} + O\left(\frac{(\log\log x)^2}{(\log x)^\eta}\right), \tag{2.2}$$

where $\eta := 1-\frac{2}{\pi}$, unless $0\leqslant f(n)\leqslant 1$ for all n in which case we take $\eta = 1-\frac{1}{\pi}$ (and adjust the power of $\log\log x$ from "2" to "3" in the error term in (2.2)).

If $1\leqslant w\leqslant (\log x)^{O(1)}$ we have, by Theorem 1.5 of [16],

$$\sum_{n\leqslant x/w} f(n) = \frac{1}{w^{1+it}}\sum_{n\leqslant x} f(n) + O\left(\frac{x}{w}\frac{(\log\log x)^2}{(\log x)^\eta}\right). \tag{2.3}$$

For a Dirichlet character $\chi \pmod q$ we define

$$S_f(x,\chi) := \sum_{n\leqslant x} f(n)\overline{\chi}(n),$$

and let $S_f(x) = S_f(x,1) = \sum_{n\leqslant x} f(n)$. We deduce the following from (2.3):

Lemma 1. *If $\psi \pmod r$ induces $\chi \pmod q$ and $q,\ell\leqslant Q_1$ then, for $t=t_{f\overline{\psi}}(x,T)$,*

$$\sum_{n\leqslant x/\ell} f(n)\overline{\chi}(n) = \frac{I(x,0,t)}{\ell^{1+it}}\prod_{p\mid q}\left(1-\frac{f(p)\overline{\psi}(p)}{p^{1+it}}\right)\sum_{n\leqslant x}\frac{f(n)\overline{\psi}(n)}{n^{it}} + O\left(\frac{q/r}{\phi(q/r)}\,\frac{x(\log\log x)^2}{\ell(\log x)^\eta}\right).$$

Proof. Let $q_r = \prod_{p\mid q, p\nmid r} p$. We have the identity

$$\begin{aligned} S_f(x/\ell,\chi) = \sum_{\substack{n\leqslant x/\ell\\(n,q_r)=1}} (f\overline{\psi})(n) &= \sum_{d\mid q_r}\mu(d)\sum_{\substack{n\leqslant x/\ell\\ d\mid n}}(f\overline{\psi})(n)\\ &= \sum_{d\mid q_r}\mu(d)(f\overline{\psi})(d)S_f(x/d\ell,\psi). \end{aligned}$$

By (2.3) and then (2.2), we have

$$S_f(x/d\ell,\psi) = \frac{1}{(d\ell)^{1+it}}\cdot I(x,0,t)\sum_{n\leqslant x} f(n)\overline{\psi}(n)n^{-it} + O\left(\frac{x}{d\ell}\,\frac{(\log\log x)^2}{(\log x)^\eta}\right).$$

Substituting this in above yields the claim. □

2.1. Mean values of multiplicative functions in arithmetic progressions.

If $(a,q)=1$ we have the usual decomposition for a sequence in arithmetic progression:

$$\sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}} f(n) = \frac{1}{\phi(q)}\sum_{\chi \pmod q}\chi(a)S_f(x,\chi). \tag{2.4}$$

To determine the largest summands on the right-hand side of (2.4) for a range of x, define

$$s_f(X,\chi) := \max_{\sqrt{X}\leqslant x\leqslant X^2}\frac{|S_f(x,\chi)|}{x},$$

and then order the characters mod q as $\chi_1, \chi_2, \dots$ so that

$$s_f(X, \chi_1) \geqslant s_f(X, \chi_2) \geqslant s_f(X, \chi_3) \geqslant \dots$$

The first part of Theorem 1.8, together with Theorem 1.9 from [16] then implies that if $q \leqslant Q_1^2$ then for any fixed $J \geqslant 2$, and all x in the range $\sqrt{X} \leqslant x \leqslant X^2$ we have

$$\sum_{\substack{\chi \pmod q \\ \chi \notin \{\chi_1, \dots, \chi_{J-1}\}}} |S_f(x, \chi)|^2 \ll_J \Big(\frac{x (\log\log x)^2}{(\log x)^{1-\frac{1}{\sqrt{J}}}} \Big)^2. \tag{2.5}$$

The proof of Theorem 1.8 from [16] then deduces from (2.4) and (2.5) that

$$\sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} f(n) = \frac{1}{\phi(q)} \sum_{j=1}^{J-1} \chi_j(a) S_f(x, \chi_j) + O\left(\frac{x}{\phi(q)} \frac{(\log\log x)^2}{(\log x)^{1-\frac{1}{\sqrt{J}}}} \right). \tag{2.6}$$

Let $\psi_j \pmod{r_j}$ be the primitive character that induces χ_j for each $j \geqslant 1$. Let $t_j = t_{f\overline{\psi}_j}(x, \log x)$ and define the multiplicative function

$$f_j(n) := f(n)\overline{\psi}_j(n) n^{-it_j} \in \mathcal{M} \text{ for each } j.$$

(We will sometimes suppress the subscript "j" and write f_* in place of f_j.) Therefore by Lemma 1 with $\ell = 1$, and taking $J \geqslant 3$ in (2.6) (as $1 - \frac{1}{\sqrt{3}} > \eta$) we obtain

$$\sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} f(n) = \frac{1}{\phi(q)} \sum_{j=1}^{J-1} \psi_j(a) k_j(q/r_j)\, I(x, t_j)\, S_{f_j}(x) + O\left(\frac{x}{\phi(q)} \frac{(\log\log x)^{2+o(1)}}{(\log x)^{\eta}} \right) \tag{2.7}$$

where k_j is the multiplicative function with $k_j(p^a) := 1 - f_j(p)/p$.

3. A certain twisted integral

We need estimates for $I(x, \beta, t) := \frac{1}{x} \int_0^x \mathrm{e}(\beta v) v^{it} \mathrm{d}v$. Evidently

$$I(x, 0, t) = \frac{x^{it}}{1 + it} \text{ and } I(x, \beta, 0) = \frac{\mathrm{e}(\beta x) - 1}{2i\pi\beta x},$$

and every $|I(x, \beta, t)| \leqslant 1$ as $|\mathrm{e}(\beta v) v^{it}| = 1$. To bound $I(x, \beta, t)$ in general we use the stationary phase method, writing $I(x, \beta, t) = \frac{1}{x} \int_0^x e^{iF(v)} \mathrm{d}v$ where $F(v) = 2\pi\beta v + t \log v$ is a real thrice-differentiable function for all $v > 0$. For any interval $0 \leqslant a < b \leqslant x$, Lemmas 4.2, 4.4 and 4.6 of [24] imply that

$$\int_a^b e^{iF(v)} \mathrm{d}v \ll \frac{1}{\min_{v \in [a,b]} |F'(v)|}, \ \frac{1}{\min_{v \in [a,b]} |F''(v)|^{1/2}}, \text{ and } \frac{1 + \sqrt{|t|}}{|\beta|},$$

respectively where, in the third case, we must have $b \ll a$ with $c := \frac{-t}{2\pi\beta} \in [a, b]$ (here c is selected so that $F'(c) = 0$). The second inequality always gives $\ll 1/\sqrt{|t|}$. If β and t have the same sign or $|t| > 3\pi|\beta|x$ then the first inequality yields $I(x, \beta, t) \ll 1/|t| \ll 1/|\beta|x$. Otherwise we use the third inequality for the interval $[\frac{c}{2}, \min\{x, 2c\}]$, and the first inequality for the rest of $[0, x]$. Collecting this together implies that

$$|I(x, \beta, t)| \ll \min\left\{ 1, \frac{1}{\sqrt{|t|}}, \ \frac{1 + \sqrt{|t|}}{|\beta|x} \right\} \ll \frac{1}{\sqrt{1 + |\beta|x}}. \tag{3.1}$$

Taking $v = xw$ and $\gamma = x\beta$ in the definition of $I(x, \beta, t)$, we obtain

$$I(x, \beta, t) = x^{it} I(1, \gamma, t) \text{ where } I(1, \gamma, t) = \int_0^1 \mathrm{e}(\gamma w) w^{it} \mathrm{d}w = \widehat{h}_t(\gamma)$$

with $h_t(w) = w^{it}$ for $0 \leqslant w \leqslant 1$, and $h_t(w) = 0$ otherwise. This implies that

$$x\int_{\beta=-\Delta/x}^{\Delta/x} |I(x,\beta,t)|^2 \mathrm{d}\beta = \int_{\gamma=-\Delta}^{\Delta} |\widehat{h}(\gamma)|^2 \mathrm{d}\gamma$$

By Plancherel's Theorem, we see this is bounded by 1 since

$$\int_{\gamma=-\infty}^{\infty} |\widehat{h}(\gamma)|^2 \mathrm{d}\gamma = \int_{w=-\infty}^{\infty} |h(w)|^2 \mathrm{d}w = 1.$$

By (3.1) we have $|\widehat{h}(\gamma)|^2 \ll (1+|t|)/\gamma^2$ and so

$$x\int_{\beta=-\Delta/x}^{\Delta/x} |I(x,\beta,t)|^2 \mathrm{d}\beta = \int_{\gamma=-\Delta}^{\Delta} |\widehat{h}(\gamma)|^2 = 1 + O\Big(\frac{1+|t|}{\Delta}\Big). \tag{3.2}$$

4. Exponential sums with multiplicative coefficients

If $(b,q)=1$ then

$$\mathrm{e}(b/q) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi}(b) g(\chi) \tag{4.1}$$

where we define the *Gauss sum* to be

$$g(\chi) = \sum_{m=1}^{q-1} \chi(m) \mathrm{e}\left(\frac{m}{q}\right).$$

A complication arises when $(b,q) > 1$. For example if q is prime then $b \equiv 0 \pmod q$ and $\mathrm{e}(b/q) = 1$ (as q is prime), a quite different formula. This explains the main technical difficulty in proving our main theorem:

Theorem 4. *Let $f \in \mathcal{M}$ and $\alpha = a/q + \beta$ where $(a,q)=1$ with $q \leqslant Q_1$, defined as in (1.8), with all the assumptions and notation given above. For any integer $J \geqslant 2$ we have*

$$\sum_{n\leqslant x} f(n)\mathrm{e}(n\alpha) = \frac{1}{\phi(q)} \sum_{\substack{j=1\\ r_j|q}}^{J-1} \overline{\psi}_j(a) g(\psi_j)\ \kappa_j(q/r_j) I(x,\beta,t_j)\cdot S_{f_j}(x) + O\big((1+|\beta|x)\mathrm{Err}_J(x,q)\big),$$

where throughout we have the error term

$$\mathrm{Err}_J(x,q) := x\frac{q}{\phi(q)}(\log\log x)^2 \left(\frac{1}{(\log x)^{1-\frac{1}{\sqrt{J}}}} + \frac{1}{\sqrt{q}(\log x)^{\eta}}\right). \tag{4.2}$$

It is worth noting that, explicitly,

$$\kappa_j(p^b) := \begin{cases} f(p^b)/p^{ibt_j} & \text{if } p|r_j; \\ \psi_j(p^b)(f_j(p^b) - f_j(p^{b-1})) & \text{if } p\nmid r_j. \end{cases} \tag{4.3}$$

The structure of the main terms in Theorem 4 bear much in common with those in (2.7). Given ψ_j, t_j, the only part of the jth summand on the right side in Theorem 4 involving the values of $f(p)$ with $p \nmid q$ is $S_{f_j}(x)$, which is independent of α.

If $|\beta| \leqslant \frac{1}{x}\log q \log\log q$ then we can simplify the error term since then

$$(1+|\beta|x)\mathrm{Err}_J(x,q) \ll x(\log\log x)^{3+o(1)} \left(\frac{1}{(\log x)^{1-\frac{1}{\sqrt{J}}}} + \frac{1}{\sqrt{q}(\log x)^{\eta}}\right).$$

For each main term we have the upper bound

$$\ll \frac{1}{\phi(q)} \cdot 1 \cdot \sqrt{r} \cdot 2^{\omega(q)-\omega(r)} \cdot \frac{1}{\sqrt{1+|\beta|x}} \cdot \frac{\phi(r)}{r} x$$
$$= \frac{1}{\sqrt{q/r}} \prod_{p|q, p\nmid r} \frac{2}{1-1/p} \cdot \frac{x}{\sqrt{q(1+|\beta|x)}} \ll \frac{x}{\sqrt{q(1+|\beta|x)}},$$

where $\omega(q)$ denotes the number of distinct prime factors of q. which is why we propose the refined Conjecture in (1.3). Taking integer $J > 1/\epsilon^2$ we deduce that if (1.3) fails then $(\log x)^{2-o(1)} \leqslant q \leqslant (\log x)^2 (\log\log x)^{1+o(1)}$, as claimed in (1.9).

4.1. **Evaluating $R_f(\alpha, x)$ for α rational: Proof of Theorem 4 when $\beta = 0$.** Writing each integer n as $m\frac{q}{d}$ where $(m, d) = 1$ we have

$$R_f(a/q, x) = \sum_{d|q} f(q/d) \sum_{\substack{m \leqslant dx/q \\ (m,d)=1}} f(m) \mathrm{e}\left(\frac{am}{d}\right) = \sum_{d|q} f(q/d) \sum_{(b,d)=1} \mathrm{e}\left(\frac{ab}{d}\right) \sum_{\substack{m \leqslant dx/q \\ m \equiv b \pmod d}} f(m).$$

We evaluate the last sum using (2.6). As $x > q^2$ the error terms add up to

$$\ll \sum_{d|q} \sum_{(b,d)=1} \frac{dx}{q\phi(d)} \frac{(\log\log x)^2}{(\log x)^{1-\frac{1}{\sqrt{J}}}} \ll \frac{q}{\phi(q)} \frac{(\log\log x)^2}{(\log x)^{1-\frac{1}{\sqrt{J}}}}.$$

For each fixed d dividing q the jth terms in (2.6) add up to

$$\frac{S_f(dx/q, \chi_j)}{\phi(d)} \cdot \sum_{(b,d)=1} \chi_j(b) \mathrm{e}\left(\frac{ab}{d}\right) = \overline{\chi}_j(a) g(\chi_j) \frac{S_f(dx/q, \chi_j)}{\phi(d)},$$

where $g(\chi_j) = g(\psi_j)\psi_j(\frac{d}{r_j})\mu(\frac{d}{r_j})$. Lemma 1 (with $q = d$ and $\ell = q/d$) implies that

$$S_f(dx/q, \chi_j) = \frac{I(x, 0, t_j)}{(q/d)^{1+it_j}} k_j(d) S_{f_j}(x) + O\left(\frac{d/r_j}{\phi(d/r_j)} \frac{x(\log\log x)^2}{(q/d)(\log x)^\eta}\right).$$

Therefore the contribution from ψ_j equals, writing $d = kr_j$, $I(x, 0, t_j) S_{f_j}(x)$ times

$$\frac{\overline{\psi}_j(a) g(\psi_j)}{\phi(r_j)} \sum_{\substack{k | \frac{q}{r_j} \\ (k, r_j)=1}} \frac{f(q/kr_j)}{(q/kr_j)^{1+it_j}} \frac{\mu(k)\psi_j(k)k_j(k)}{\phi(k)} = \frac{\overline{\psi}_j(a) g(\psi_j)}{\phi(q)} \kappa\left(\frac{q}{r_j}\right)$$

since $k(r_j) = 1$, plus an error term of

$$\ll \frac{r_j}{\phi(r_j)} \frac{1}{\sqrt{q/r_j}} \sum_{\substack{k | \frac{q}{r_j} \\ (k, r_j)=1}} \frac{\mu^2(k)k^2}{\phi(k)^2} \frac{x(\log\log x)^2}{\sqrt{q}(\log x)^\eta} \ll \frac{q}{\phi(q)} \frac{x(\log\log x)^2}{\sqrt{q}(\log x)^\eta}.$$

This yields Theorem 4 when $\beta = 0$. □

4.2. **Evaluating $R_f(\alpha, x)$ for α irrational: Proof of Theorem 4 when $\beta \neq 0$.** The starting point is the identity

$$R_f(x, \alpha) = \mathrm{e}(\beta x) R_f(x, a/q) - 2\pi i\beta \int_1^x \mathrm{e}(\beta v) R_f(v, a/q) \mathrm{d}v. \tag{4.4}$$

We truncate the integral at $\frac{x}{\log x}$ at a cost of $\leqslant 2\pi|\beta|\int_1^{x/\log x} v\mathrm{d}v = \pi|\beta|x\cdot\frac{x}{(\log x)^2}$. We then substitute in the formula for when $\beta = 0$ which we established in the precious subsection. Integrating the error term, we obtain the total error

$$O\left((1+|\beta|x)\mathrm{Err}_J(x,q)\right),$$

with the notation (4.2). The jth term becomes $\frac{1}{\phi(q)}\overline{\psi}_j(a)g(\psi_j)\kappa_j(\frac{q}{r_j})I(x,0,t_j)$ times

$$\mathrm{e}(\beta x)\sum_{n\leqslant x} f_j(n) - 2\pi i\beta\int_{x/\log x}^{x}\mathrm{e}(\beta v)(v/x)^{it_j}\sum_{n\leqslant v} f_j(n)\mathrm{d}v.$$

By (2.3) this has main term

$$\left\{\mathrm{e}(\beta x) - 2\pi i\beta\int_{x/\log x}^{x}\frac{\mathrm{e}(\beta v)}{(x/v)^{1+it_j}}\mathrm{d}v\right\}\sum_{n\leqslant x} f_j(n)$$

plus the error term

$$\ll 2\pi|\beta|\int_{x/\log x}^{x} v\frac{(\log\log v)^2}{(\log v)^\eta}\mathrm{d}v \ll |\beta|x^2\frac{(\log\log x)^2}{(\log x)^\eta}.$$

Extending the integral to 0 yields an error term $\ll |\beta|\int_0^{x/\log x}\frac{v}{x}\mathrm{d}v\cdot x \leqslant |\beta|x\cdot\frac{x}{(\log x)^2}$. Then, integrating by parts, we obtain

$$\mathrm{e}(\beta x) - 2\pi i\beta\int_0^x\frac{\mathrm{e}(\beta v)}{(x/v)^{1+it_j}}\mathrm{d}v = \frac{I(x,\beta,t_j)}{I(x,0,t_j)}.$$

The result follows by substituting this in above. □

5. Twisting by periodic functions

If $h(n)$ is a function of period q then for any Dirichlet character $\psi \pmod r$ where r divides q, and any integer D for which $r|D|q$ define the *pseudo-Gauss sums*

$$G_\psi(D) := \sum_{a=1}^{D}\psi(a)h\left(\frac{aq}{D}\right)$$

We modify the argument of section 4.1 to prove the following general result.

Theorem 5. *Let $h(n)$ be a function of period q with $q\leqslant(\log x)^{O(1)}$. Let $f\in\mathcal{M}$ and define the ψ_j as before. Fix $\epsilon>0$. Then*

$$\sum_{n\leqslant x} f(n)h(n) = \frac{1}{\phi(q)}\sum_{\substack{j=1\\ r_j|q}}^{J-1}\left(\sum_{r_j|n|q} k_j(n)\kappa_j\left(\tfrac{q}{n}\right)G_{\psi_j}(n)\right)I(x,0,t_j)S_{f_j}(x)$$
$$+ O\left(\frac{1}{q}\sum_{m=0}^{q-1}|h(m)|\cdot\frac{x}{(\log x)^{1-\epsilon}} + \frac{1}{q}\sum_{\substack{j=1\\ r_j|q}}^{J-1}\sum_{r_j|n|q} 2^{\omega(q/n)}|G_{\psi_j}(n)|\cdot\frac{x}{(\log x)^{\eta-\epsilon}}\right).$$

The function h enters into the main term in Theorem 5 only within the pseudo-Gauss sums $G_\psi(\cdot)$. In usual Gauss sums one can always reduce to the case where ψ and h have the same period (via some simple identities). We cannot do that here so perhaps we should use a different definition for pseudo-Gauss sums ? For example we can rewrite the

main term here: By expanding the $G_{\psi_j}(n)$ and re-organizing, the parenthesized part of the jth main term can be written as

$$\frac{1}{q}\sum_{r_j|m|q}\frac{m}{\phi(m)}\frac{f(q/m)}{(q/m)^{it}}\,k_j((m,\tfrac{q}{m}))G^\dagger_{\psi_j}(m) \text{ where } G^\dagger_{\psi_j}(m):=\sum_{(b,m)=1}\psi_j(b)h\Big(\frac{bq}{m}\Big).$$

Proof. We again use our estimates of f in arithmetic progressions:

$$\sum_{n\leqslant x}f(n)h(n)=\sum_{d|q}f\left(\frac{q}{d}\right)\sum_{(b,d)=1}h\left(\frac{bq}{d}\right)\sum_{\substack{m\leqslant dx/q\\ m\equiv b \pmod d}}f(m).$$

We evaluate the last sum using (2.6). The error terms add up to

$$\ll\sum_{d|q}\sum_{(b,d)=1}\left|h\left(\frac{bq}{d}\right)\right|\frac{dx}{q\phi(d)}\,\frac{(\log\log x)^2}{(\log x)^{1-\frac{1}{\sqrt{J}}}}\ll\frac{1}{q}\sum_{m=0}^{q-1}|h(m)|\cdot\frac{x}{(\log x)^{1-\epsilon}}.$$

taking any integer $J>1/\epsilon^2$. For each fixed d dividing q the jth terms in (2.6) add up to

$$\frac{S_f(dx/q,\chi_j)}{\phi(d)}\cdot\sum_{(b,d)=1}\chi_j(b)h\left(\frac{bq}{d}\right)=\frac{S_f(dx/q,\chi_j)}{\phi(d)}\sum_{\substack{\ell|d\\(\ell,r_j)=1}}\mu(\ell)\psi_j(\ell)G_{\psi_j}(d/\ell).$$

Lemma 1 (with $q=d$ and $\ell=q/d$) implies that

$$S_f(dx/q,\chi_j)=\frac{I(x,0,t_j)}{(q/d)^{1+it_j}}k_j(d)S_{f_j}(x)+O\left(\frac{d/r_j}{\phi(d/r_j)}\,\frac{x(\log\log x)^2}{(q/d)(\log x)^\eta}\right).$$

Therefore the contribution from ψ_j equals, writing $d=kr_j, k=\ell m$ and $n=mr_j$, $I(x,0,t_j)S_{f_j}(x)$ times

$$\sum_{m|\frac{q}{r_j}}G_{\psi_j}(mr_j)\sum_{\ell|\frac{q}{mr_j}}\frac{f(q/\ell mr_j)}{(q/\ell mr_j)^{1+it_j}}\frac{k_j(\ell m)}{\phi(\ell mr_j)}\mu(\ell)\psi_j(\ell)=\frac{1}{\phi(q)}\sum_{r_j|n|q}G_{\psi_j}(n)k_j(n)\kappa_j\left(\frac{q}{n}\right)$$

We also have an error term of

$$\ll\frac{1}{q}\sum_{r_j|n|q}|G_{\psi_j}(n)|\sum_{\substack{\ell|q/n\\(\ell,r_j)=1}}\mu^2(\ell)\frac{\ell n}{\phi(\ell n)}\frac{\ell n/r_j}{\phi(\ell n/r_j)}\,\frac{x(\log\log x)^2}{(\log x)^\eta}$$

$$\ll\frac{1}{q}\sum_{r_j|n|q}|G_{\psi_j}(n)|2^{\omega(q/n)}\,\frac{x(\log\log x)^{2+o(1)}}{(\log x)^\eta}$$

Collecting together these estimates gives the Theorem. □

5.1. **Periodic functions of size one.** In this subsection we will restrict our attention to periodic functions h, of minimal period q, with the property that

$$h(n)=\prod_{p^e\|q}h_p(n)$$

where each h_p has minimal period p^e. Examples include characters mod q and exponentials like $\mathrm{e}(\frac{g(n)}{q})$ where $g(x)\in\mathbb{Z}[x]$ or $g(n)=an+b\overline{n}$ defined only if $(n,q)=1$, where $\overline{n}$ is the inverse of $n \pmod q$. Moreover $h(.)$ might be the product of such functions, like

$$h(n):=\chi_1(n+a_1)\cdots\chi_m(n+a_m)\mathrm{e}(\tfrac{g(n)}{q})$$

where we are given characters $\chi_1, \dots, \chi_m \pmod q$ and integers $a_1, \dots, a_m$ for some $m \geqslant 1$, with $g(\cdot)$ as above. For each of these cases, Theorem 6 of Chapter 6 in [20] gives that

$$\Big| \sum_{n \pmod{p^e}} h_p(n) \Big| \leqslant (m+d)\, p^{e/2}$$

for all prime powers p^e, and so by the the Chinese Remainder Theorem

$$\Big| \sum_{n (\mathrm{mod}\ q)} h(n) \Big| \leqslant (m+d)^{\omega(q)} q^{1/2}.$$

We can also apply this result for $\psi(n)h(n)$, where ψ has conductor dividing q, provided this also has minimal period q.

We wish to apply Theorem 5 to these $h(.)$. We restrict attention to q squarefree (to avoid the case where p divides both r and q/D). In this case we can deduce the bounds $|G_\psi(D)| \leqslant (m+d)^{\omega(D)} D^{1/2}$; and the error term in Theorem 5 becomes

$$\ll \frac{x}{(\log x)^{1-\epsilon}} + \frac{x}{\sqrt{q}(\log x)^{\eta-\epsilon}}.$$

Let $h_m(n) = \prod_{p^e \| m} h_p(n)$ whenever $m|q$. If $(D, q/D) = (r, D/r) = 1$ (which always happens if q is squarefree) then by the Chinese Remainder Theorem we have

$$G_\psi(D) = h_{q/D}(0)\overline{\psi}(\tfrac{q}{D}) \cdot \prod_{p^e \| r} \sum_{a=1}^{p^e} \psi_p(a)h(a) \cdot \prod_{p^f \| D/r} \sum_{b=1}^{p^f} h_p(b)$$

where $\psi = \prod_{p^e \| r} \psi_p$ and ψ_p is a character mod p^e. Therefore the coefficients in Theorem 5 become, if $(r, q/r) = 1$ and q/r is squarefree, writing $f^*(p)$ in place of $f_j(p)$ (and suppressing the subscript j everywhere)

$$\sum_{r|n|q} k(n)\kappa\big(\tfrac{q}{n}\big) G_\psi(n) = \prod_{p^e \| r} \sum_{a=1}^{p^e} \psi_p(a)h(a) \cdot \prod_{p | \frac{q}{r}} \Big((1-\tfrac{1}{p}) h_p(0) f^*(p) + (1 - \tfrac{f^*(p)}{p}) \sum_{a=1}^{p-1} h(a) \Big)$$

Taking absolute values, in the above examples, this is

$$\leqslant \prod_{p^e \| r} (m+d)\, p^{e/2} \prod_{p | \frac{q}{r}} (p-1)(1+\tfrac{2}{p}) \leqslant (m+d)^{\omega(r)} \frac{q}{\phi(q)} \frac{q}{\sqrt{r}}$$

Therefore we deduce that

$$\sum_{n \leqslant x} f(n)h(n) = \sum_{\substack{j=1 \\ r_j | q}}^{J-1} \frac{c_j}{\sqrt{r_j}} \frac{x^{it_j}}{1+it_j} S_{f_j}(x) + O\Big(\frac{x}{(\log x)^{1-\epsilon}} + \frac{x}{\sqrt{q}(\log x)^{\eta-\epsilon}} \Big), \tag{5.1}$$

where each $|c_j| \leqslant (m+d)^{\omega(r_j)} \frac{q^2}{\phi(q)^2}$.

6. Development of the circle method

We will apply Theorem 4 to applications of the circle method using multiplicative functions.

6.1. **Major and minor arcs.** Let $Q = x/(\log x)^{\tau-\epsilon}$ where $\tau = \frac{2-\sqrt{2}}{3}$, and for each α on the unit circle we select $q \leqslant Q$ such that $|\alpha - a/q| \leqslant 1/qQ$ for some integer a coprime with q. Define the major arcs $\mathfrak{M}$ by

$$\mathfrak{M} = \bigcup_{\substack{(a,q)=1\\ q\leqslant x/Q}} \left[\frac{a}{q} - \frac{1}{qQ},\ \frac{a}{q} + \frac{1}{qQ}\right],$$

and let $\mathfrak{m}$ the minor arcs defined by $[0,1) \smallsetminus \mathfrak{M}$, where $[0,1)$ stands for $\mathbb{R}/\mathbb{Z}$.

Lemma 2. *With the notation as above (but with $\psi = \psi_1, \kappa = \kappa_1, r = r_1, t = t_1, f_* = f_1$ we can write*

$$R_f(\alpha, x) = M_f(\alpha, x) + E_f(\alpha, x)$$

where $M_f(\alpha) = M_f(\alpha, x) = 0$ if $\alpha \in \mathfrak{m}$ and

$$M_f(\alpha) = \frac{\overline{\psi}(a) g(\psi) 1_{r|q}\, \kappa(\frac{q}{r})}{\phi(q)} I(x, \beta, t) \cdot S_{f_*}(x) \qquad \text{if } \alpha \in \mathfrak{M},$$

with

$$E_f(\alpha) = E_f(\alpha, x) \ll \frac{x}{(\log x)^{\tau/2 - o(1)}} \qquad \text{for all } \alpha. \tag{6.1}$$

We also have that

$$\int_0^1 |R_f(\alpha, x)|^2 \mathrm{d}\alpha,\ \int_0^1 |M_f(\alpha)|^2 \mathrm{d}\alpha,\ \int_0^1 |E_f(\alpha)|^2 \mathrm{d}\alpha \ll x.$$

Proof. We get better estimates for the minor arcs than claimed here, from (1.7). Taking $J = 2$ in Theorem 4 we obtain the main term $M_f(\alpha, x)$, with

$$E_f(\alpha, x) \ll \left(1 + \frac{x}{qQ}\right) \frac{x}{(\log x)^{\frac{3}{2}\tau - o(1)}} \ll \frac{x}{(\log x)^{\frac{\tau}{2} + \epsilon - o(1)}}$$

Therefore

$$\begin{aligned}\int_{\mathfrak{M}} |E_f(\alpha)|^2 &= \sum_{q\leqslant x/Q} \sum_{(a,q)=1} \int_{-1/qQ}^{1/qQ} |E_f(\alpha + a/q)|^2 \mathrm{d}\alpha \\ &\ll \sum_{q\leqslant x/Q} \frac{\phi(q)}{qQ} \frac{x^2}{q^2Q^2} \frac{x^2}{(\log x)^{2-\sqrt{2}-o(1)}} \\ &\ll \frac{x^4}{Q^3 (\log x)^{2-\sqrt{2}-o(1)}} \ll \frac{x}{(\log x)^{3\epsilon - o(1)}} = o(x).\end{aligned}$$

Now, if $\alpha \in \mathfrak{m}$ then $x/Q \leqslant q \leqslant Q$ and so we may take $R = x/Q$ in (1.7) to obtain

$$M_f(\alpha) = 0 \text{ and } E_f(\alpha) \ll \frac{x}{(\log x)^{\frac{1}{2}(\tau-\epsilon) - o(1)}} \text{ if } \alpha \in \mathfrak{m}.$$

Letting $\epsilon \to 0$ we deduce (6.1). By Parseval we have

$$\int_0^1 |R_f(\alpha, x)|^2 \mathrm{d}\alpha = \sum_{n\leqslant x} |f(n)|^2 \leqslant x,$$

and so

$$\int_0^1 |M_f(\alpha)|^2 \mathrm{d}\alpha = \int_{\mathfrak{M}} |M_f(\alpha)|^2 \mathrm{d}\alpha \leqslant 2 \int_{\mathfrak{M}} (|R_f(\alpha, x)|^2 + |E_f(\alpha)|^2) \mathrm{d}\alpha \ll x,$$

and therefore

$$\int_0^1 |E_f(\alpha)|^2 \mathrm{d}\alpha = \int_{\mathfrak{M}} |E_f(\alpha)|^2 \mathrm{d}\alpha + \int_{\mathfrak{m}} |R_f(\alpha, x)|^2 \mathrm{d}\alpha \ll x,$$

as desired. □

6.2. **Mean value of a multiplicative function on a weighted set of integers.** Given $f \in \mathcal{M}$ define, for convenience, $S_f(x) =: x\mu(f,x)$ where $\mu(f,x) := \frac{1}{x}\sum_{n\leqslant x} f(n)$. The "structure theorem" of [17] states that for a given $f \in \mathcal{M}$,

$$\mu(f,x) = \mu(f^{(s)},x)\mu(f^{(\ell)},x) + O\left(\frac{(\log\log x)^{1+2\eta}}{(\log x)^{\eta}}\right). \tag{6.2}$$

where $f^{(s)}, f^{(\ell)}$ are multiplicative functions with $f = f^{(s)}f^{(\ell)}$ defined by

$$f^{(s)}(p) = \begin{cases} f(p)p^{-it} & \text{if } p \leqslant z \\ 1 & \text{if } p > z \end{cases} \quad \text{and} \quad f^{(\ell)}(p) = \begin{cases} p^{it} & \text{if } p \leqslant z \\ f(p) & \text{if } p > z \end{cases}, \tag{6.3}$$

where $t = t_f(x, \log x)$ and $z = (\log x)^A$ for some constant $A > 0$. Here we similarly define the two multiplicative functions $F_s, F_\ell \in \mathcal{M}$ with $F_sF_\ell = f$ as follows:

$$F_s(p) = \begin{cases} f(p) & \text{for } p \leqslant z, \\ \psi(p)p^{it} & \text{for } p > z, \end{cases} \quad \text{and} \quad F_\ell(p) = \begin{cases} 1 & \text{for } p \leqslant z, \\ f_*(p) := f(p)\overline{\psi}(p)p^{-it} & \text{for } p > z. \end{cases}$$

We note that $(f_*)^{(s)} = (F_s)_*$ and $(f_*)^{(\ell)} = F_\ell = (F_\ell)_*$.

We now use Theorem 4 to obtain an estimate for the sum over $n \leqslant x$, of $f(n)$ times an arbitrary weight w_n.

Proposition 1. *If $f \in \mathcal{M}$ and $\{w_n\}_{n\leqslant x}$ is a set of weights then*

$$\sum_{n\leqslant x} w_n f(n) = \frac{1}{x}\sum_{n\leqslant x} F_\ell(n) \cdot \sum_{n\leqslant x} w_n F_s(n) + O\Big(\|W\|_1 \frac{x}{(\log x)^{\tau/2-o(1)}}\Big) \tag{6.4}$$

where $g = f_1$, $W(t) = \sum_{n\leqslant x} w_n \mathrm{e}(-nt)$ and $\|W\|_1 := \int_0^1 |W(\alpha)|\mathrm{d}\alpha$.

Proof. By Plancherel's theorem we have

$$\sum_{n\leqslant x} w_n f(n) = \int_0^1 R_f(\alpha, x)W(\alpha)\mathrm{d}\alpha$$

and, by (6.1), this is

$$\int_0^1 M_f(\alpha)W(\alpha)\mathrm{d}\alpha + O\left(\|W\|_1 \frac{x}{(\log x)^{\tau/2-o(1)}}\right).$$

Now $M_f(\alpha) = 0$ if $\alpha \in \mathfrak{m}$, and so by Lemma 2

$$\int_0^1 M_f(\alpha)W(\alpha)\mathrm{d}\alpha = \sigma(f)S_{f_*}(x),$$

where $S_{f_*}(x) := \sum_{n\leqslant x} f_*(n)$ and

$$\sigma(f) = \sum_{\substack{q\leqslant x/Q \\ r|q}} \sum_{(a,q)=1} \frac{\overline{\psi}(a)g(\psi)\kappa(\frac{q}{r})}{\phi(q)} \int_{-1/qQ}^{1/qQ} I(x,\beta,t)W(a/q+\beta)\mathrm{d}\beta.$$

Therefore

$$\sum_{n\leqslant x} w_n f(n) = \sigma(f)S_{f_*}(x) + O\left(\|W\|_1 \frac{x}{(\log x)^{\tau/2-o(1)}}\right).$$

Now (6.2) and the discussion that follows it, implies that

$$S_{f_*}(x) = \mu(f_*, x)x = \mu((F_s)_*, x)\mu(F_\ell, x)x + O\left(\frac{x}{(\log x)^{\eta - o(1)}}\right).$$

Moreover the only term in the summands for $\sigma(f)$ that directly involves values of f is $\kappa(q/r)$, and this involves only $f(p)$ for primes $p \leqslant x/Q$. Now $f(p) = F_s(p)$ for all $p \leqslant x/Q$ (as $x/Q \leqslant z$) and, by definition, $\psi_f = \psi_{F_s}$ and $t_f(x, \log x) = t_{F_s}(x, \log x)$. Therefore

$$\sigma(f) = \sigma(F_s),$$

and so

$$\sum_{n \leqslant x} w_n f(n) = \sigma(F_s)\mu((F_s)_*, x)\mu(F_\ell, x)x + O\left(\|W\|_1 \frac{x}{(\log x)^{\tau/2 - o(1)}}\right),$$

since $|\sigma(f)| \ll \|W\|_1$ by definition and $\tau/2 < \eta$. By the same argument we also have

$$\sum_{n} w_n F_s(n) = \sigma(F_s)\mu((F_s)_*, x)x + O\left(\|W\|_1 \frac{x}{(\log x)^{\tau/2 - o(1)}}\right),$$

and (6.4) follows by comparing the last two displayed equations, as $|\mu(F_\ell, x)| \leqslant 1$. □

6.3. **Applications.**

Theorem 6. *Let $f \in \mathcal{M}$ and $z = \log x$. If $A, B \subset \{1, 2, \dots, x\}$ with $A + B \subset \{1, 2, \dots, x\}$, then*

$$\frac{1}{|A||B|} \sum_{\substack{a \in A \\ b \in B}} f(a+b) = \frac{1}{x} \sum_{n \leqslant x} F_\ell(n) \cdot \frac{1}{|A||B|} \sum_{\substack{a \in A \\ b \in B}} F_s(a+b) + O\left(\frac{x}{(|A||B|)^{1/2}(\log x)^{\tau/2 - o(1)}}\right).$$

The effect of the large primes is independent of the particular choice of sets A and B

Proof. Let

$$w_n = \frac{1}{|A||B|} \#\{(a, b) \in A \times B \,:\, n = a + b\},$$

and proceed as above by noting that $W(\alpha) = A(\alpha)B(\alpha)/|A||B|$ so that

$$\begin{aligned} \|W\|_1 = \int_0^1 |W(\alpha)| \mathrm{d}\alpha &= \frac{1}{|A||B|} \int_0^1 |A(\alpha)B(\alpha)| \mathrm{d}\alpha \\ &\leqslant \frac{1}{|A||B|} \left(\int_0^1 |A(\alpha)|^2 \mathrm{d}\alpha \int_0^1 |B(\alpha)|^2 \mathrm{d}\alpha\right)^{1/2} = \frac{1}{(|A||B|)^{1/2}}, \end{aligned}$$

which, together with (6.4), implies the result. □

6.4. **Explicit Theorem 6: The mean value of $F_s(a+b)$ over $a \in A,\ b \in B$.**

Lemma 3. *With the notation as in Theorem 6, we have*

$$\frac{1}{|A||B|} \sum_{\substack{a \in A \\ b \in B}} F_s(a+b) = \sum_{P(m) \leqslant z} \kappa(m) \cdot \frac{1}{|A||B|} \sum_{\substack{a \in A \\ b \in B \\ m \mid a+b}} \psi\left(\frac{a+b}{m}\right)(a+b)^{it}$$

Proof. If χ_z the characteristic function of z-friable integers then we have identity

$$F_s(n) = n^{it}((\kappa\chi_z) * \psi)(n).$$

We deduce that

$$\sum_n w_n F_s(n) = \sum_{P(m) \leqslant z} \kappa(m) \sum_{n:\ m \mid n} w_n n^{it} \psi(n/m), \tag{6.5}$$

which gives the result. □

It is also worth observing that by inclusion-exclusion we have

$$\frac{1}{x}\sum_{n\leqslant x}F_\ell(n)=\prod_{p\leqslant z}\frac{(1-f_*(p)/p)}{(1-1/p)}\cdot\frac{1}{x}\sum_{n\leqslant x}f_*(n)+O\left(\frac{1}{(\log x)}\right).$$

Together with Lemma 3 this allows us to give a more explicit main term in Theorem 6.

7. Three term products in the circle method

7.1. **Proof sketch of the formulas of Theorems 2 and 3.** We have

$$\sum_{\substack{\ell,m,n\leqslant x\\ a\ell+bm+cn=0}}f(\ell)g(m)h(n)=\sum_{\ell,m,n\leqslant x}f(\ell)g(m)h(n)\int_0^1\mathrm{e}(t(a\ell+bm+cn))\mathrm{d}t$$
$$=\int_0^1R_f(at,x)R_g(bt,x)R_h(ct,x)\mathrm{d}t,$$

and, similarly,

$$\sum_{\substack{\ell,m,n\geqslant 1\\ \ell+m+n=N}}f(\ell)g(m)h(n)=\int_0^1\mathrm{e}(-Nt)R_f(t)R_g(t)R_h(t)\mathrm{d}t \tag{7.1}$$

We decompose the integrands using the identity

$$R_f(\alpha,x)R_g(\beta,x)R_h(\gamma,x)-M_f(\alpha)M_g(\beta)M_h(\gamma)$$
$$=R_f(\alpha,x)R_g(\beta,x)E_h(\gamma)+R_f(\alpha,x)E_g(\beta)M_h(\gamma)+E_f(\alpha)M_g(\beta)M_h(\gamma).$$

For any functions $A(\alpha)$ and $B(\alpha)$ we have

$$\left|\int_0^1A(\alpha)B(\alpha)E(\alpha)\mathrm{d}\alpha\right|\leqslant\max_\alpha|E(\alpha)|\int_0^1|A(\alpha)B(\alpha)|\mathrm{d}\alpha$$
$$\leqslant\max_\alpha|E(\alpha)|\int_0^1\big(|A(\alpha)|^2+|B(\alpha)|^2\big)\mathrm{d}\alpha$$

Therefore by the estimates of Lemma 2 we deduce that each of

$$\int_0^1R_f(at,x)R_g(bt,x)E_h(ct)\mathrm{d}t,\ \int_0^1R_f(at,x)E_g(bt)M_h(ct)\mathrm{d}t\text{ and }\int_0^1E_f(at)M_g(bt)M_h(ct)\mathrm{d}t$$

are $\ll x^2/(\log x)^{\tau/2-o(1)}$. Therefore

$$\sum_{\substack{\ell,m,n\leqslant x\\ a\ell+bm+cn=0}}f(\ell)g(m)h(n)=\int_0^1M_f(at)M_g(bt)M_h(ct)\mathrm{d}t+O\left(\frac{x^2}{(\log x)^{\tau/2-o(1)}}\right),$$

and

$$\sum_{\substack{\ell,m,n\geqslant 1\\ \ell+m+n=N}}f(\ell)g(m)h(n)=\int_0^1\mathrm{e}(-Nt)M_f(t)M_g(t)M_h(t)\mathrm{d}t+O\left(\frac{x^2}{(\log x)^{\tau/2-o(1)}}\right).$$

From here we imitate the proof of Proposition 1: First we replace M_f, M_g and M_h by the relevant expressions given in Lemma 2 and the main term becomes an integral over the major arcs. The main term is the product of the mean values of f_*, g_* and h_*, times an expression that depends only on the values of $f(p), g(p)$ and $h(p)$ with $p\leqslant z$. Therefore if we compare this with the same calculation, but now working with F_s, G_s, H_s we discover that the (complicated) main term is the same in each case, other than now

we have the product of the mean values of $(F_s)_*, (G_s)_*$ and $(H_s)_*$. We can compensate for this by multiplying through by the product of the mean values of $(F_\ell)_* = F_\ell, (G_\ell)_* = G_\ell$ and $(H_\ell)_* = H_\ell$. We deduce the formulas in the first parts of Theorems 2 and 3. □

8. Explicit evaluation of the mean value of $F_s(\ell)G_s(m)H_s(n)$

In this section we complete the proofs of Theorem 2 and 3.

8.1. **The mean value of $F_s(\ell)G_s(m)H_s(n)$ over solutions to $a\ell + bm + cn = 0$.** In this subsection we will sketch a proof that

$$\frac{1}{x^2/2} \sum_{\substack{\ell,m,n\leqslant x\\ a\ell+bm+cn=0}} F_s(\ell)G_s(m)H_s(n) = 2\mathcal{E}(\infty)x^{it}\ \delta_{f,g,h}\ \prod_{p\leqslant z}\mathcal{E}(p) + O\Big(\frac{1}{\log x}\Big) \tag{8.1}$$

where $t = t_f + t_g + t_h$, the factor $\delta_{f,g,h} = 1$ if $\psi_f\psi_g\psi_h$ is principal, and 0 otherwise,

$$\mathcal{E}(\infty) = \frac{1}{|c|}\int_{\substack{0\leqslant u,v,w\leqslant 1\\ au+bv+cw=0}} u^{it_f}v^{it_g}w^{it_h}\,\mathrm{d}u\mathrm{d}v,$$

the $\mathcal{E}(p)$ are appropriate local factors for each prime $p \leqslant z$.

The main idea is to replace each $F_s(\ell)$ (as well as $G_s(m)$ and $H_s(n)$) by a periodic function times ℓ^{it_f}, and so evaluate them separately. Let e_p be the smallest integer for which $p^{e_p} > z^2$, and $N = r_f r_g r_h \prod_{p\leqslant z} p^{e_p}$.

If v is the largest z-smooth divisor of n then $F_s(n) = f_*(v)\psi(n)n^{it}$. Define $f_\dagger(p^k) := f_*(p^{\min\{k,e_p\}})\psi(p^k)$ for all $p \leqslant z$, and $f_\dagger(p^k) := \psi(p^k)$ if $p > z$, so that $F_s(n) = f_\dagger(n)n^{it}$ provided v divides N. The number of $n \leqslant x$ for which $F_s(n) \neq f_\dagger(n)n^{it}$ is

$$\leqslant \sum_{p\leqslant z}\frac{x}{p^{e_p}} < \frac{x\pi(z)}{z^2} < \frac{x}{z}.$$

Therefore the accumulated error term in our sum by this change is $\ll x^2/z$ which is acceptable.

Now if $u \equiv n \pmod N$ then $\psi(n) = \psi(u)$ as r_ψ divides N and if $f_\dagger = f_\dagger^*\psi_f$ then $f_\dagger^*$ only depends on the primes dividing N and so $f_\dagger^*(n) = f_\dagger^*((n,N)) = f_\dagger^*((u,N)) = f_\dagger^*(u)$. Therefore each $f_\dagger(n) = f_\dagger(u)$ and so the sum in (8.1) is equal to

$$\sum_{\substack{u,v,w \pmod N\\ au+bv+cw\equiv 0 \pmod N}} f_\dagger(u)g_\dagger(v)h_\dagger(w) \sum_{\substack{\ell,m,n\leqslant x\\ a\ell+bm+cn=0\\ \ell\equiv u,\ m\equiv v,\ n\equiv w \pmod N}} \ell^{it_f}m^{it_g}n^{it_h} + O\Big(\frac{x^2}{\log x}\Big).$$

Since a, b, c are pairwise coprime, we see that the inner sums are all non-empty. Given one term ℓ_0, m_0, n_0 in the inner sum, the others all look like $\ell_0 + \alpha N, m_0 + \beta N, n_0 + \gamma N$ where $a\alpha+b\beta+c\gamma = 0$. We deduce that the outer sum is $(x/N)^2\mathcal{E}(\infty)x^{it}(1+O(1/\log x))$. Therefore the left-hand side of (8.1) equals

$$2\mathcal{E}(\infty)x^{it}\cdot\frac{1}{N^2}\sum_{\substack{u,v,w \pmod N\\ au+bv+cw\equiv 0 \pmod N}} f_\dagger(u)g_\dagger(v)h_\dagger(w) + O\Big(\frac{1}{\log x}\Big).$$

The sum remains unchanged if we multiply u, v and w through by any reduced residue $t \pmod N$. This changes each summand by a factor $(\psi_f\psi_g\psi_h)(t)$. This implies that either the sum is 0, or $(\psi_f\psi_g\psi_h)(t) = 1$ for all such t, in which case $\psi_f\psi_g\psi_h$ is principal.

The inner sum can be made into an Euler product by the Chinese Remainder Theorem, and therefore we obtain (8.1). To be more precise about the Euler factors $\mathcal{E}(p)$ in the case that $p \nmid abcr_fr_gr_h$:

$$\begin{aligned}\mathcal{E}(p) &= \frac{1}{p^{2e}} \sum_{\substack{u,v,w \pmod{p^e} \\ au+bv+cw\equiv 0 \pmod{p^e}}} f_\dagger((u,p^e))g_\dagger((v,p^e))h_\dagger((w,p^e)) \\ &= \frac{1}{p^2}(p^2-3p+2+(p-1)(f_\dagger(p)+g_\dagger(p)+h_\dagger(p))+O(1)) \\ &= 1-\frac{1-f_\dagger(p)}{p}-\frac{1-g_\dagger(p)}{p}-\frac{1-h_\dagger(p)}{p}+O\Big(\frac{1}{p^2}\Big)\end{aligned}$$

since the probability that p^2 divides one of u, v, w, or p divides two of u, v, w is $O(1/p^2)$. We deduce that

$$\prod_{p\leqslant z}\mathcal{E}(p) \asymp \exp\left(-\sum_{p\leqslant z}\frac{1-f_\dagger(p)}{p}-\sum_{p\leqslant z}\frac{1-g_\dagger(p)}{p}-\sum_{p\leqslant z}\frac{1-h_\dagger(p)}{p}\right).$$

Now letting $z\to\infty$ we see that if this converges then ψ_f, ψ_g, ψ_h are each principal, and so $\psi_f = \psi_g = \psi_h = 1$ since they are all primitive characters. This completes the proof of Theorem 2.

The exactly analogous argument completes the proof of Theorem 3.

8.2. **When f, g and h are real-valued in Theorems 2 and 3.** If we have a non-zero mean value in Theorem 2 or 3 then $\psi_f = \psi_g = \psi_h = 1$ so that if $p \leqslant z$ then $f_\dagger(p) = f_*(p) = f(p)p^{-it_f}$. Therefore if f is real-valued and $\prod_{p\leqslant z}\mathcal{E}(p) \gg 1$ then $\sum_{p\leqslant z}\frac{1-f_\dagger(p)}{p} \ll 1$ and so $t_f = 0$ (and similarly $t_g = t_h = 0$). Thus

$$\mathcal{E}(\infty)x^{it} = \frac{1}{|c|}\int_{\substack{0\leqslant u,v,w\leqslant 1\\ au+bv+cw=0}} \mathrm{d}u\,\mathrm{d}v,$$

in Theorem 2 (which equals $\frac{1}{2}$ if $a=b=-c=1$), and $2\mathcal{E}(\infty)x^{it}=1$ in Theorem 3.

When $a=b=1, c=-1$ the Euler product term is

$$\begin{aligned}\mathcal{E}(p) &= \lim_{e\to\infty}\frac{1}{p^{2e}}\sum_{u+v\equiv w \pmod{p^e}} f((u,p^e))g((v,p^e))h((w,p^e)) \\ &=: \mathcal{E}^*(p)\cdot\Big(1-\frac{1}{p}\Big)^3\Big(1-\frac{f(p)}{p}\Big)^{-1}\Big(1-\frac{g(p)}{p}\Big)^{-1}\cdot\Big(1-\frac{h(p)}{p}\Big)^{-1},\end{aligned}$$

say. Now $F_s = f^{(s)}$ and $F_\ell = f^{(\ell)}$, and

$$\sum_{n\leqslant x} f_s(n) \sim \prod_{p\leqslant z}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1}\cdot x,$$

by inclusion-exclusion, so that by using (6.2) we have

$$\prod_{p\leqslant z}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1}\sum_{n\leqslant x}F_\ell(n) \sim \sum_{n\leqslant x} f(n).$$

Substituting this all back into Theorem 2 we obtain

$$\frac{1}{x^2/2}\sum_{\substack{\ell,m,n\leqslant x\\ \ell+m=n}} f(\ell)g(m)h(n) = \mu(f,x)\mu(g,x)\mu(h,x)\prod_{p\leqslant z}\mathcal{E}^*(p)+o(1). \tag{8.2}$$

If say $h(p)=1$ then

$$\mathcal{E}(p)=\lim_{e\to\infty}\frac{1}{p^e}\sum_{u \pmod{p^e}} f((u,p^e))\cdot\frac{1}{p^e}\sum_{u \pmod{p^e}} g((v,p^e))$$
$$=\Big(1-\frac{1}{p}\Big)^2\Big(1-\frac{f(p)}{p}\Big)^{-1}\Big(1-\frac{g(p)}{p}\Big)^{-1},$$

and so $\mathcal{E}^*(p)=1$. Therefore if $\mathcal{E}^*(p)\neq 1$ then $f(p),g(p),h(p)\neq 1$.

If $f(p)=g(p)=h(p)=-1$ then

$$\mathcal{E}^*(p):=\Big(1-\frac{8}{(p-1)^2}\Big(1+\frac{1}{p^2}\Big)^{-1}\Big).$$

Therefore the right-hand side of (8.2) becomes

$$\mu(f,x)\mu(g,x)\mu(h,x)\prod_{\substack{p\leqslant z\\ f(p)=g(p)=h(p)=-1}}\Big(1-\frac{8p^2}{(p-1)^2(p^2+1)}\Big)+o(1).$$

Remark. This is $\sim \mu(f,x)\mu(g,x)\mu(h,x)$ if and only if $\mu(f,x)\mu(g,x)\mu(h,x)\gg 1$ and $P:=\{p\leqslant z:\ f(p)=g(p)=h(p)=-1\}=\emptyset$: To see this suppose $P\neq\emptyset$. If odd $p\in P$ then $|\mathcal{E}^*(p)|<1$ and so $\prod_{p\in P}|\mathcal{E}^*(p)|<1$ unless $2\in P$. But then $\prod_{p\in P}|\mathcal{E}^*(p)|\geqslant \prod_p|1-\frac{8p^2}{(p-1)^2(p^2+1)}|=1.322\ldots>1$,

If f,g,h only take values 0 or 1 (with $a=b=1,c=-1$) then

$$\frac{1}{x^2/2}\sum_{\substack{\ell,m,n\leqslant x\\ \ell+m=n}} f(\ell)g(m)h(n)=\mu(f,x)\mu(g,x)\mu(h,x)\prod_{\substack{p\leqslant z\\ f(p)=g(p)=h(p)=0}}\Big(1-\frac{1}{(p-1)^2}\Big)+o(1),$$

from which we immediately deduce (1.5).

We can proceed analogously in Theorem 3, obtaining

$$\frac{1}{x^2/2}\sum_{\substack{\ell,m,n\leqslant x\\ \ell+m+n=N}} f(\ell)g(m)h(n)=\mu(f,x)\mu(g,x)\mu(h,x)\prod_{p\leqslant z}\mathcal{E}_N^*(p),+o(1)$$

for appropriate Euler factors $\mathcal{E}_N^*(p)$, and again we note that $\mathcal{E}_N^*(p)=1$ if $f(p),g(p)$ or $h(p)=1$. If $f(p)=g(p)=h(p)=0$ then

$$\mathcal{E}_N^*(p)=\Big(1-\frac{1}{p}\Big)^{-3}\frac{1}{p^2}\sum_{\substack{u+v+w\equiv N \pmod p\\ (uvw,p)=1}}1=\begin{cases}1+\frac{1}{(p-1)^3} & \text{if } p\nmid N;\\ 1-\frac{1}{(p-1)^2} & \text{if } p|N,\end{cases}$$

which implies (1.6). If $f(p)=g(p)=h(p)=-1$ then one can similarly calculate a formula for $\mathcal{E}_N^*(p)$ but it appears to be complicated.

8.3. **Sketch of proof of Corollary 2.** The density of solutions to $f(a)=g(b)=h(c)=-1$ with $a+b=c\leqslant x$ is

$$\frac{1}{x^2/2}\sum_{\substack{a,b,c\leqslant x\\ a+b=c}}\tfrac{1}{8}(1-f(a))(1-g(b))(1-h(c)). \tag{8.3}$$

When we expand this we have the sum of eight mean values and we can apply (8.2) to them all, noting that $\mathcal{E}^*(p)=1$ except in the term with the product $f(a)g(b)h(c)$. Therefore if the mean values of f,g and h are δ_f,δ_g and δ_h, respectively, then (8.3) is

$$\tfrac{1}{8}\left((1-\delta_f)(1-\delta_g)(1-\delta_h)+(1-C_{\mathcal{P}})\delta_f\delta_g\delta_h\right)+o(1) \tag{8.4}$$

where

$$C_{\mathcal{P}} := \prod_{p\in\mathcal{P}} \Big(1 - \frac{8p^2}{(p-1)^2(p^2+1)}\Big)$$

with $\mathcal{P} := \{p : f(p) = g(p) = h(p) = -1\}$, by (8.2). We need to maximize the expression in (8.4):

By the main result of [12], we have $\delta_f, \delta_g, \delta_h \in [\delta_0\alpha_{\mathcal{P}}, \alpha_{\mathcal{P}}]$ where $\alpha_{\mathcal{P}} = \prod_{p\in\mathcal{P}} \frac{p-1}{p+1}$.

Now $|(1-C_{\mathcal{P}})\delta_1\delta_2\delta_3| \leqslant |1-C_{\mathcal{P}}|\alpha_{\mathcal{P}}^3 \leqslant \frac{32}{135}$ which is attained with $\mathcal{P} = \{2\}$. If (8.4) is $> \kappa$ then $\min(1-\delta_f, 1-\delta_g, 1-\delta_h) > 1.57$, so each of $\delta_f, \delta_g, \delta_h$ are negative. We can then show that the expression in (8.4) is maximized, (considering the cases where $1 - C_{\mathcal{P}}$ is positive or negative separately) when $\delta_f = \delta_g = \delta_h = -\delta_0$.

For the second part of Corollary 2, we must minimize

$$\tfrac{1}{8}\left((1+\delta_f)(1+\delta_g)(1+\delta_h) + (C_{\mathcal{P}}-1)\delta_f\delta_g\delta_h\right) + o(1). \tag{8.5}$$

In the smallest solution there cannot be two positive δ's, else we just replace them both by their negative. If just one δ is positive then $(C_{\mathcal{P}}-1)\delta_1\delta_2\delta_3 < 0$, else replacing δ_i by $-\delta_i$ makes both terms smaller. But then $C_{\mathcal{P}} < 1$ and thus we optimize when $\mathcal{P} = \{2\}$. Therefore the quantity inside the brackets in (8.5) is $\geqslant (1-\frac{1}{3}\delta_0)^2 - (\frac{32}{5\cdot 3^3}\delta_0^2 > \frac{1}{2} > (1-\delta_0)^3$. Hence all the δ's must be negative, and if (8.5) is $> \frac{1}{8}(1-\delta_0)^3$ then $C_{\mathcal{P}} > 1$. But then $2, 3 \in C_{\mathcal{P}}$, and so the quantity inside the brackets in (8.5) is $\geqslant (1-\delta_0/6)^3 - \frac{83}{25\cdot 6^3}\delta_0^3 > 0.7 > (1-\delta_0)^3$. Therefore the minimum occurs when $\delta_1 = \delta_2 = \delta_3 = -\delta_0$. □

9. Binary additive problems

The granddaddy of all additive problems in number theory is, of course, the Goldbach problem. Typically any problem involving two variables linked by a linear equation is considered very difficult to deal with. However Brüdern [3] has developed a method that works in many situations provided certain hypotheses are fulfilled (see also Theorem 1.9 in [18] which uses techniques related to those used here). We shall now investigate Brüdern's idea in our context. As usual we evaluate

$$\sum_{a+b=N} f(a)g(b) = \int_0^1 \mathrm{e}(-\alpha N)R_f(\alpha, N)R_g(\alpha, N)\mathrm{d}\alpha \tag{9.1}$$

where a, b are positive integers, and f and g are multiplicative functions taking values inside or on the unit circle, using the circle method. The hope is that the main part of the integral on the right side lies on the major arcs, and the contribution on the minor arcs is easily proved to be negligible (that is $o(N)$). Let $\mathfrak{M}\cup\mathfrak{m}$ be the partition of $[0,1)$ into major and minor arcs, respectively. Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\int_{\mathfrak{m}} \mathrm{e}(-\alpha N)R_f(\alpha, N)R_g(\alpha, N)\mathrm{d}\alpha\right|^2 &\leqslant \int_{\mathfrak{m}} |R_f(\alpha, N)|^2\mathrm{d}\alpha \cdot \int_{\mathfrak{m}} |R_g(\alpha, N)|^2\mathrm{d}\alpha \\ &\leqslant N\int_{\mathfrak{m}} |R_f(\alpha, N)|^2\mathrm{d}\alpha\end{aligned}$$

since $\int_0^1 |R_g(\alpha, N)|^2\mathrm{d}\alpha = \sum_{n\leqslant N} |g(n)|^2$. Hence the contribution of the minor arcs in (9.1) is $o(N)$ if $\int_{\mathfrak{m}} |R_f(\alpha, N)|^2\mathrm{d}\alpha = o(N)$.

Proposition 2. *Suppose that $f \in \mathcal{M}$ and $|f(n)| = 1$ for all integers $n \geqslant 1$. Then we have $\int_{\mathfrak{m}} |R_f(\alpha, x)|^2\mathrm{d}\alpha = o(x)$ if and only if $\int_{\mathfrak{M}} |M_f(\alpha)|^2\mathrm{d}\alpha = x + o(x)$. We always have*

$\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha \leqslant x+o(x)$. *Moreover* $\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha = x+o(x)$ *if and only if*

$$\sum_{(x/Q)^\epsilon<p\leqslant x}\frac{1-\Re e(f(p)\overline{\psi}(p)p^{-it})}{p}=o(1). \tag{9.2}$$

Proof. During the proof of Lemma 2 we saw that

$$\int_{\mathfrak{M}}|E_f(\alpha)|^2\mathrm{d}\alpha = o(x) \text{ and } \int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha \ll x,$$

so that

$$\Big|\int_{\mathfrak{M}} M_f(\alpha)\overline{E_f(\alpha)}\mathrm{d}\alpha\Big| \leqslant \Big(\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha\int_{\mathfrak{M}}|E_f(\alpha)|^2\mathrm{d}\alpha\Big)^{1/2} = o(x),$$

and therefore

$$\begin{aligned}\int_{\mathfrak{M}}|R_f(\alpha,x)|^2\mathrm{d}\alpha - \int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha&\\ \leqslant 2\left|\int_{\mathfrak{M}}|M_f(\alpha)\overline{E_f(\alpha)}\mathrm{d}\alpha\right| + \int_{\mathfrak{M}}|E_f(\alpha)|^2\mathrm{d}\alpha &= o(x).\end{aligned}$$

By Parseval we have

$$\int_0^1|R_f(\alpha,x)|^2\mathrm{d}\alpha = \sum_{n\leqslant x}|f(n)|^2 = x+O(1),$$

and so

$$\int_{\mathfrak{m}}|R_f(\alpha,x)|^2\mathrm{d}\alpha = o(x) \text{ if and only if } \int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha = x+o(x).$$

Now, by Lemma 2, we have

$$\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha = |\mu(f_*,x)|^2x^2\sum_{r|q\leqslant x/Q}r\frac{|\kappa(q/r)|^2}{\phi(q)}\int_{\beta=-1/qQ}^{1/qQ}|I(x,\beta,t)|^2\mathrm{d}\beta.$$

If $q\ll x/Q$ then $x\int_{\beta=-1/qQ}^{1/qQ}|I(x,\beta,t)|^2\mathrm{d}\beta \asymp 1$ by (3.2); and it is $1+o(1)$ if $q=o(x/Q)$. Using this and (6.2) we deduce that

$$\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha \lesssim |\mu((f_*)_s,x)|^2|\mu((f_*)_\ell,x)|^2\,x\sum_{r|q\leqslant x/Q}r\frac{|\kappa(q/r)|^2}{\phi(q)}$$

with, say, $z=x/Q$. Since $|\kappa(m))| = \prod_{p|m}|1-f_*(p)|$ this last sum is

$$\leqslant r\sum_{\substack{m\geqslant 1\\ p|m\Longrightarrow p\leqslant x/Q}}\frac{|\kappa(m)|^2}{\phi(rm)} = \prod_{p\leqslant x/Q}\Big(1-\frac1p\Big)^{-2}\Big|1-\frac{f_*(p)}{p}\Big|^2 \sim \frac{1}{|\mu((f_*)_s,x)|^2}.$$

Now since $|\mu((f_*)_\ell,x)|\leqslant 1$, we deduce that $\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha \lesssim x$. In order to get equality here we must have (asymptotic) equality in each of the last few steps. In particular we must have $|\mu((f_*)_\ell,x)|\sim 1$, and so

$$\sum_{x/Q<p\leqslant x}\frac{1-\Re e(f_*(p))}{p}=o(1)$$

by Halász's Theorem. In this case we deduce that $\int_{\mathfrak{M}}|M_f(\alpha)|^2\mathrm{d}\alpha\sim x$ if and only if

$$\sum_{1\leqslant m=o(x/Qr)}\frac{r}{\phi(mr)}\prod_{p|m}|1-f_*(p)|^2 \sim \frac{1}{|\mu((f_*)_s,x)|^2},$$

since the larger $q = mr$ in the sum are weighted by an I-integral that is < 1. One can show that this happens if and only if the primes $> (x/Q)^\epsilon$ do not make a significant contribution; that is,

$$\sum_{(x/Q)^\epsilon < p \leqslant x/Q} \frac{1 - \Re e(f_*(p))}{p} = o(1)$$

□

As a corollary we can recover the hypothesis (9.3) used by Brüdern [3]:

Corollary 3. *Suppose that $f \in \mathcal{M}$ and $|f(n)| = 1$ for all integers $n \geqslant 1$. Then we have $\int_{\mathfrak{m}} |R_f(\alpha, x)|^2 \mathrm{d}\alpha = o(x)$ for all sufficiently large x if and only if*

$$\sum_p \frac{1 - \Re e(f(p)\overline{\psi}(p)p^{-it})}{p} \ll 1 \tag{9.3}$$

If (9.2) holds then we can evaluate the main term in certain binary additive problems, like (9.1) to obtain

$$\sum_{a+b=N} f(a)g(b) = \int_{\mathfrak{M}} \mathrm{e}(-\alpha N) M_f(\alpha, N) M_g(\alpha, N) \mathrm{d}\alpha + o(N) \tag{9.4}$$

or

$$\sum_{n \leqslant N} f(n)g(n+1) = \int_{\mathfrak{M}} \mathrm{e}(\alpha) M_f(\alpha, N) M_g(-\alpha, N) \mathrm{d}\alpha + o(N) \tag{9.5}$$

using only the major arcs. The actual asymptotics can be obtained without using the circle method, as in [18], Corollary 1.4, so we will not give explicit details here.

10. Upper bounds on exponential sums

In this section we give explicit upper bounds on $R_f(\alpha, x)$.

10.1. **The conjecture** (1.2) **can only fail in very specific circumstances.** We sketch a proof of the following:

Proposition 3. *If* (1.2) *does not hold then* (1.9) *holds.*

Proof Sketch. Let $\varepsilon > 0$ be fixed. Let Q_1 be defined by (1.8) and $Q_2 = x/(\log x)^3$. There exists $q \leqslant Q_2$ such that $|\alpha - a/q| \leqslant 1/qQ_2$ for some integer a.

If $Q_1 < q \leqslant Q_2$ then (1.2) follows from (1.7).

If $q \leqslant Q_1$ and $|\alpha - a/q| > (\log q \log\log q)/x =: 1/qQ_3$, then there exists $r \leqslant Q_3$ such that $|\alpha - b/r| \leqslant 1/rQ_3$ for some integer b, and hence $r \neq q$. Therefore, as $Q_3 > 2q$,

$$\frac{1}{qQ_2} + \frac{1}{2qr} > \frac{1}{qQ_2} + \frac{1}{rQ_3} \geqslant |\alpha - a/q| + |\alpha - b/r| \geqslant |a/q - b/r| \geqslant \frac{1}{qr},$$

which implies that $r > Q_2/2 > Q_1$ and then (1.2) follows from (1.7).

If $q \leqslant Q_1$ and $|\alpha - a/q| \leqslant (\log q \log\log q)/x = 1/qQ_3$ then we apply Theorem 4. As $S_{f_j}(x) \ll (\phi(r)/r)x$, the jth term is

$$\ll \prod_{p^a \| q, p \nmid r} \frac{2p}{(p-1)p^{a/2}} \prod_{p^a \| q,\ p^b \| r} \frac{p^{b/2}}{p^{a/2}} \cdot \frac{x}{\sqrt{q}} \ll \frac{x}{\sqrt{q}}.$$

Moreover in this range the error term is, taking $J > 1/\varepsilon^2$,

$$\ll \frac{x}{(\log x)^{1-\varepsilon}} + \frac{x}{\sqrt{q}}.$$

This is $\ll x/\sqrt{q}$ for $q \leqslant (\log x)^{2-\varepsilon}$. □

10.2. **Exponential sums over friable numbers.** We sketch the minor changes needed for the following slightly stronger estimate than Proposition 1 of [7]: If $|\alpha - a/q| \leqslant 1/q^2$ with $(a,q)=1$ and $q \leqslant x$ then

$$\sum_{\substack{n\leqslant x\\ P(n)\leqslant y}} f(n)\mathrm{e}(\alpha n) \ll \sqrt{xy} + \left(\frac{x}{\sqrt{q}} + \sqrt{xq\log(2x/q)}\right)\log y + \frac{x}{\mathrm{e}^{\{\frac{1}{2}+o(1)\}\sqrt{\log x\log\log x}}}. \tag{10.1}$$

Proof. We follow the proof of Proposition 1 of [7] (though rectify the omission of the $h=0$ term at the end of the long display of equations at the top of page 62), but now choosing $T = \exp(\sqrt{\log x \log\log x})$. The key change that we make is to give the (easily proved) more precise upper bound

$$\sum_{1\leqslant h\leqslant x/K} \min\left(K, \frac{1}{\|\theta h\|}\right) \ll K + q\log(2x/q) + \frac{x}{q} + \frac{x}{K}\log 2K$$

(rather than the typographically simpler $\ll (K+q+x/q+x/K)\log x$). The only other change is that we note that the sum of $(xK)^{1/2}$, over K of the form 2^jT with j an integer, and $2^jT \leqslant y$, is $\ll (xy)^{1/2}$. □

It would be of interest to remove further logs from the right-hand side of (10.1): For instance, based on the case $y=x$ we believe, based on (1.7), that we should have something like $\sqrt{xy}/\log y$ in place of the $\sqrt{xy}$ term.

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Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Diderot, Sorbonne Paris Cité, UMR 7586, Case Postale 7012, F-75251 Paris CEDEX 13, France
Email address: regis.delabreteche@imj-prg.fr

Département de Mathématiques et Statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montréal, QC H3C 3J7, Canada; and Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England.
Email address: andrew@dms.umontreal.ca