How Inge Lehmann discovered the inner core of the Earth*

Christiane Rousseau, Université de Montréal

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Abstract

In this paper we explain the mathematics behind the discovery by Inge Lehmann that the inner core of the Earth is solid, using data collected around the Earth on seismic waves and their traveling time through the Earth.

1 Introduction

Preamble. *This paper belongs to the special issue of CMJ on* Mathematics of Planet Earth 2013*, of which I am the initiator and coordinator. Although I am a pure mathematician myself, I have always been fascinated by the breadth and power of mathematical tools in other sciences. One of my pleasures when working for MPE2013 is that I learn new fascinating applications on a very regular basis. I had heard that Inge Lehmann had discovered that the inner core of the Earth was solid and I wanted to understand the idea. So I decided to read her paper [2], which is fortunately available on Internet. It is not necessarily easy to read a paper in another discipline, here geophysics. Fortunately, near the end of the paper, in a very pedagogical way, Inge Lehmann sketched a very simple model of the Earth which she used to sell her discovery. Just looking at her model and figure, I was stimulated to fill in the mathematical details and missing calculations. I found them sufficiently interesting to share them with you. Meanwhile, I also discovered that Inge Lehmann had a training as a mathematician, and I invite you to learn more about her fascinating life by surfing on Internet.*

It is not an easy matter to study the structure of the interior of the Earth. We cannot dig and observe with our eyes. All observations are indirect, and what we cannot see with our eyes, we see with our *mathematical eyes*. As an example, already the Greek Erasthothenes knew that the Earth was a sphere and could estimate its radius by measuring the angle of the sun with the vertical direction at different cities (see for instance [1]). We can estimate the approximate mass of the Earth by measuring

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the gravitational force at the surface of the Earth described by Newton's gravitational law and learn from that that the inner part of the Earth has a much higher density than the surface part. Indeed, Let M be the mass of the Earth and R, its radius. Newton's gravitational law implies that a body of mass m located at the surface of the Earth is attracted by the Earth with a gravitational force of size

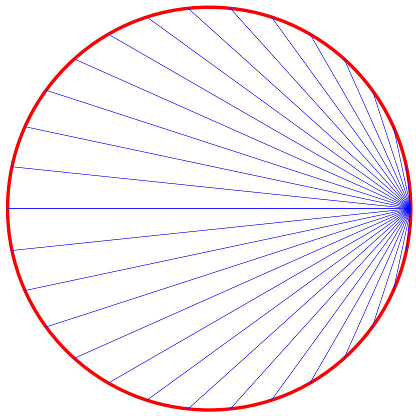
$$F = GmM\frac{1}{R^2},$$

where G is the known gravitational constant. On the other side, the size of this force is $F = mg$, where g is the acceleration at the surface of the Earth, which can be measured to be approximately $9.8\,\mathrm{m/s}$. Putting $GmM\frac{1}{R^2} = mg$ and simplifying m, we get $M = \frac{gR^2}{G}$. The mass obtained is much too large for the average density of $3,000\,\mathrm{kg/m^3}$ near the surface of the Earth, yielding that denser materials must exist at deeper levels.

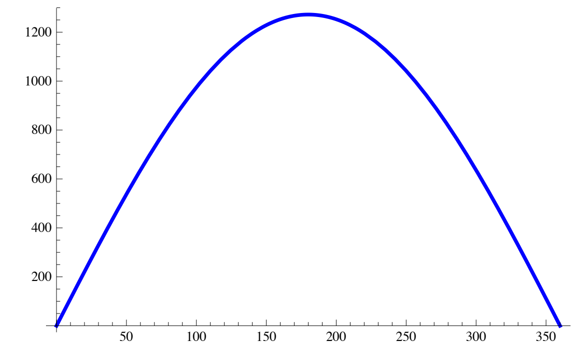
We can also explore the inner structure of the Earth by indirect methods which are part of the field of *remote sensing*. There is now a consensus among the scientists on the structure of the Earth. The theory of a liquid interior below the crust dates back to Richard Dixon Oldham at the turn of the 20th century. But it is only in 1936 that Inge Lehmann discovered the inner core. More than 30 years later, Freeman Gilbert and Adam M. Dziewonski established that the inner core is solid by other indirect types of arguments. Inge Lehmann worked for the Royal Danish Geodetic Institute. She had access to data on the seismic waves generated by major earthquakes and recorded by seismographs located at different stations around the world. To explain her reasoning in analyzing the internal structure of the Earth, we need to introduce a model of the Earth, and then refine it in several steps.

A spherical model of the Earth. Let us approximate the Earth by a sphere of radius R and make the hypothesis that its interior has spherical symmetry, i.e. it has the same structure along any ray through the center. Let us now consider an earthquake occurring at one point on the surface of the Earth and sending seismic waves in all directions. Each tangent line to a seismic wave at the earthquake together with the center of the Earth determines a plane. Because of the spherical symmetry, the seismic wave propagates inside that plane, thus allowing a planar model as in Figure 1. Let us suppose that the earthquake occurs at the point $(R, 0)$. We have a family of seismic waves starting at $(R, 0)$ and directed towards the points $(R\cos\theta, R\sin\theta)$ where $R = 6360\,\mathrm{km}$ is the radius of the Earth and $\theta \in [0, 2\pi]$. Let τ be the value of θ in degrees.

If the interior of the Earth were homogeneous, then the seismic waves would propagate along straight lines at constant speed. Figure 1(a) represents the travel paths of waves in a plane through the center of the sphere. You can check that the length of a path starting at $(R, 0)$ and ending at $(R\cos\theta, R\sin\theta)$ is given by $2R\sin\frac{\theta}{2}$. Near the crust the speed of seismic waves is approximately $10\,\mathrm{km/s}$, which yields the paths and travel times of Figure 1 for a radius of the Earth of $6360\,\mathrm{km}$. **So any deviation**



(a) The paths of the waves



(b) The travel time (in seconds) of the waves in (a) depending on the angular coordinate (in degrees) of the end point of the wave

Figure 1: The paths and travel times of the seismic waves in a homogeneous Earth for a speed of 10 km/s and a radius of the Earth of 6360 km. The travel time is a function of the angle θ: on the horizontal axis the angle is given in degrees.

from these observations tells us that the hypothesis of a uniform Earth is not valid. . . , and we need to add some refinement to the model.

Geophysics is a complex field, and there are a lot of phenomena that need to be taken into account. For instance, there are several types of seismic waves generated by a seism. In particular, two types of waves propagate inside the Earth: the P-waves are *pressure waves* which are longitudinal, and the S-waves are *shear waves* which are transversal. The S-waves do not travel in liquid. Since they are not detected far from the epicenter of an earthquake, Richard Dixon Oldham concluded that they are stopped by some liquid interior of the Earth.

We now come to the simple model proposed by Inge Lehmann. Recall that the radius of the Earth is approximately 6360 km: the Earth is a flattened ellipsoid of revolution, but we will limit ourselves to the spherical approximation. In Lehmann's simple model, the interior of the Earth has three main strata: the mantle to a depth of 2890 km, the outer core to a depth of 5150 km and, finally, the inner core. There are more strata in practice, but these do not change significantly the rough picture given by the simple model.

2 The propagation of seismic waves inside the Earth

In a non uniform Earth, the speed of propagation of a seismic wave varies. This changes also the direction of the wave, following the Snell-Descartes law of refraction:

Law of refraction. Consider a beam of light as it travels through a uniform material with speed v_1 and transitions into another uniform material where it travels with speed v_2. Let θ_1 be the angle of the beam of light through the first material, as measured from the perpendicular of the interface between the two materials. Similarly, let θ_2 be the angle of the beam of light through the second material, measured from the same perpendicular (see Figure 2). Then the law of refraction states that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

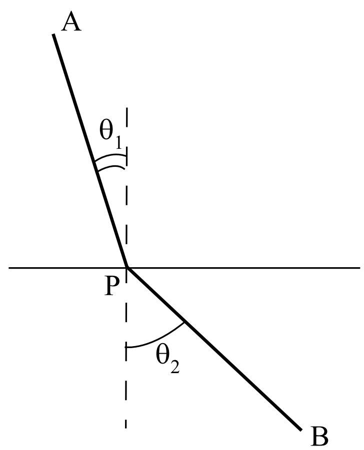


Figure 2: The law of refraction.

Hence, if θ_1, v_1 and v_2 are known we can calculate θ_2 by means of

$$\sin\theta_2 = \frac{v_2}{v_1}\sin\theta_1. \tag{1}$$

But suppose that v_2 is larger that v_1, i.e. the beam of light is coming from the slow side. If θ_1 is sufficiently large, then $\frac{v_2}{v_1}\sin\theta_1 > 1$ and (1) has no solution! What happens? The beam of light is reflected as in the law of reflection illustrated in Figure 3!

Law of reflection. As a beam of light arrives at the surface of a mirror, it is reflected such that the angle of incidence θ_1 is equal to the angle of reflection θ_2 (see Figure 3).

Fermat's principle explains this and unifies the two laws:

Fermat's principle. The path followed by a beam of light traveling from a point A to a point B minimizes the travel time from A to B.

This principle allows to compute the travel path of a beam of light in a non homogeneous media. The mathematical techniques are part of a beautiful field of mathematics called *calculus of variations*, but we will not go into these details, but the curious reader can consult the chapter on calculus of variations in [3].

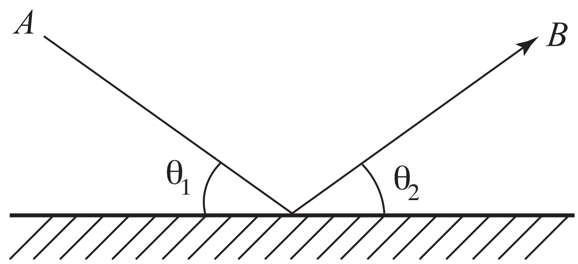


Figure 3: The law of reflection.

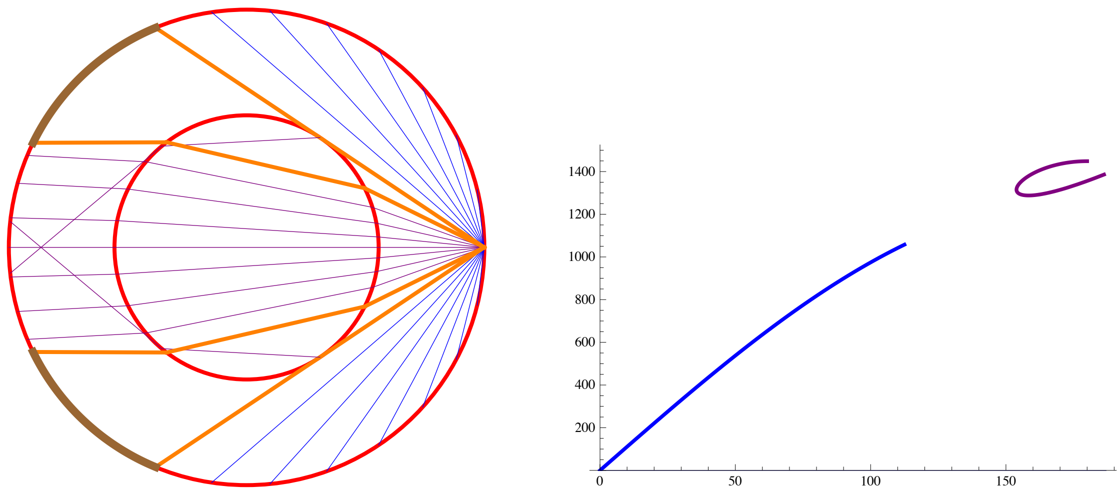
3 An Earth with a mantle and a core

At the time of Inge Lehmann it was reasonably admitted that a mantle which is approximately 2,890 km thick would surround a core. Since the radius of the Earth is approximately 6,360 km, this gives a core with radius 5/9 of that of the Earth. Following Lehmann's model, we take a speed of 10 km/h for the seismic waves in the mantle and of 8 km/h in the core, and we compute the travel paths of the seismic waves and their travel time.

We keep the hypothesis that the interior of the Earth has a spherical symmetry and we consider an earthquake occurring at one point on the surface of the Earth and sending seismic waves in all directions. As before, we can limit ourselves to the planar model which is illustrated in Figure 4. Let us suppose that the earthquake occurs at the point $(R, 0)$ and consider a seismic wave starting at $(R, 0)$ and directed towards the point $(R\cos\theta, R\sin\theta)$. For the travel time we only plot the values of $\theta \in [0, \pi]$. Let τ be the value of θ in degrees, and $\phi(\tau)$ be the angular coordinate where the wave reaches the surface of the Earth. The wave starting at $(R, 0)$ is tangent to the core if $\tau = 112^{\text{o}}$. So the waves follow straight lines for $\tau \in [0, 112^{\text{o}}]$. For a higher value of τ, the wave is refracted when entering the core. The refracted wave corresponding to $\tau = 112^{\text{o}}$ reaches the surface of the Earth for $\phi(\tau) = 186^{\text{o}}$. But refracted waves may intersect each other, and refracted waves with initial direction corresponding to $\tau \in [112^{\text{o}}, 180^{\text{o}}]$ intersect the Earth at points corresponding to $\phi(\tau) \in [154^{\text{o}}, 186^{\text{o}}]$. No wave is detected satisfying $\phi(\tau) \in [112^{\text{o}}, 154^{\text{o}}]$, and there are some values of $\phi(\tau) \in [154^{\text{o}}, 180^{\text{o}}]$, which correspond to two distinct waves with different travel times.

You are intrigued by the special form of the right part of the curve of the travel time in Figure 4? The explanation can be seen in Figure 5. We consider a wave starting at $(R, 0)$ and directed towards $R(\cos\tau, \sin\tau)$, where the angle τ (in degrees) satisfies $\tau \in [112^{\text{o}}, 180^{\text{o}}]$, i.e. the wave is refracted. Figure 5(a) presents the graph of the travel time $T(\tau)$ of this wave until it reaches the surface of the Earth at a point $R(\cos\phi(\tau), \sin\phi(\tau))$. Figure 5(b) represents the graph of $\phi(\tau)$, and Figure 5(c) is simply the parametric plot $(\phi(\tau), T(\tau))$.

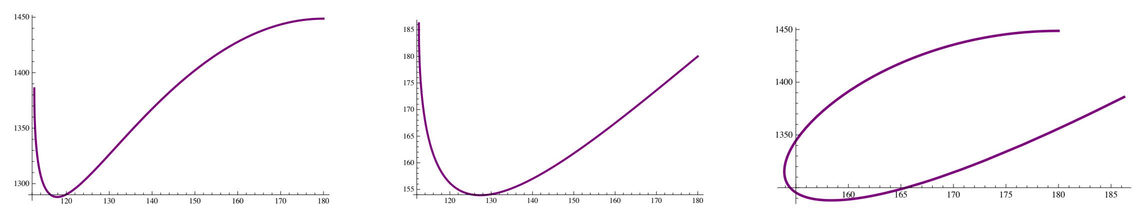
You are wondering whether some waves could be reflected on the inner sphere? This



(a) The paths of the waves.

(b) The travel time (in seconds) of the waves in (a) depending on the angular coordinate (in degrees) of the end point of the wave.

Figure 4: The paths and travel times of the seismic waves in the Earth for a speed of 10 km/s inside the mantle and of 8 km/h inside the core. Note that refracted waves can intersect each other! The special form of the right part of the curve giving the travel time of waves is explained in Figure 5 below.



(a) The graph of the travel time $T(\tau)$ as a function of τ.

(b) The exit angle $\phi(\tau)$ as a function of τ.

(c) The parametric plot $(\phi(\tau), T(\tau))$.

Figure 5: The right part of the curve giving the travel time of the refracted waves in Figure 4 is the parametric plot $(\phi(\tau), T(\tau))$ for a wave starting at $(R, 0)$ and directed towards $R(\cos\tau, \sin\tau)$, where $\tau \in [112^{\mathrm{o}}, 180^{\mathrm{o}}]$.

is not possible. Indeed, consider a wave arriving from the earthquake and making an angle θ_1 with the normal to the inner sphere as in Figure 6. Since it is on the fast side, it is necessarily refracted, and the refracted wave makes an angle θ_2 with the normal to the sphere. But it will then intersect again the inner sphere with the same angle θ_2 with the normal at the second intersection point, and hence be refracted outside, making the same angle θ_1 with the normal.

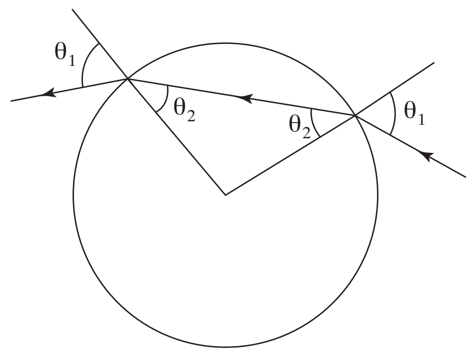
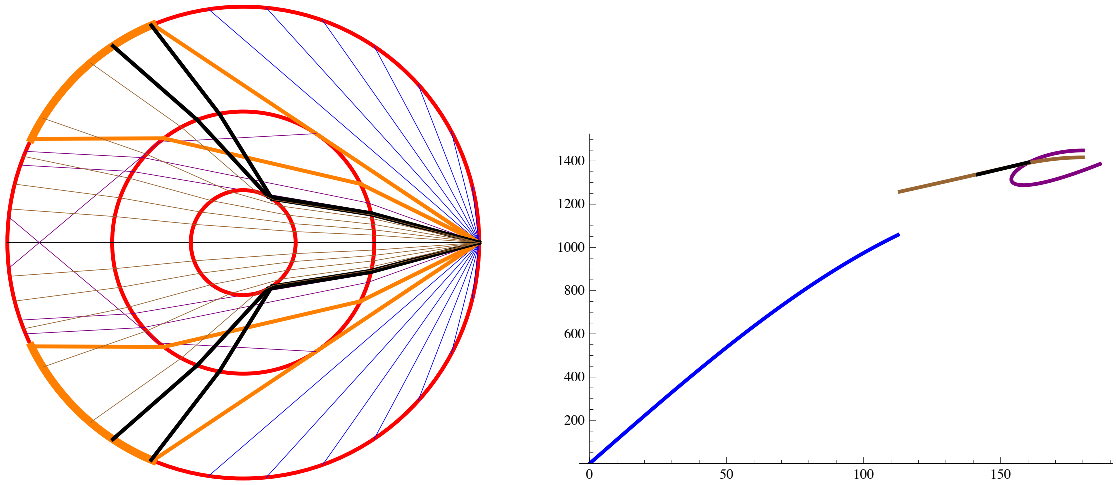


Figure 6: The symmetry of the travel path of a refracted wave through the inner sphere.

4 The inner core

But what about if we detect waves for some values of $\tau \in [112^o, 154^o]$ and measure their travel time? **This means that our model has a flaw...** and we must correct the model so that it fits with the observed data. We are now facing an *inverse problem* that has to be solved: understanding the inner structure of the Earth from the travel times of waves at different locations of the Earth. Inge Lehman did face this problem. And she deduced that the core was not homogeneous: there is rather a smaller inner core, surrounded by the outer core. The wave is traveling faster in the inner core. Hence, the wave could be reflected on the inner core if it is arriving too tangentially.

In this paper, we will limit ourselves to the direct problem and complete the figure of the model of Inge Lehmann. To the previous Figure 4 we must add an inner core whose radius is approximately 2/9 of that of the Earth, and we suppose that the travel speed of the waves in the inner core is 8.8 km/h (see Figure 7). The only work we have to do is to compute explicitly the travel paths of the different waves, depending on their departure angle. These waves can be reflected or refracted when they change layer. Each step can be easily computed with elementary Euclidean and analytic geometry, but putting all the pieces together for several waves requires some programming. For instance, the figures of the paper have been programmed on Mathematica. Instead of presenting all details we will just show the elementary steps for the planar problem that can allow you to continue and make the program yourself.



(a) The paths of the waves: the paths between the two black lines are reflected on the inner core, while the others are only refracted.

(b) The travel time (in seconds) of the waves in (a) depending on the angular coordinate (in degrees) of the end point of the wave

Figure 7: The paths and travel times of the seismic waves in the Earth for a speed of 10 km/s inside the mantle, 8 km/h inside the outer core, and 8.8 km/h inside the inner core.

4.1 Some details of the computations

You may be wondering how to draw Figure 7 and think that it is a difficult problem because you have not done similar things before? In this section, we will explain you how to make the calculations. You will probably be surprised how elementary the first steps are. And after a few of them, you will not need anymore explanations for continuing by yourself. For our figure, we start with the three circles of R, $\frac{5}{9}R$ and $\frac{2}{9}R$, that we call respectively large circle, middle circle, and small circle. To simplify, we can of course suppose that $R = 1$. We also suppose that the epicenter of the earthquake is located at the point $(1, 0)$.

Computing the waves and their intersection with the circles. We will need to work with the equation of lines supporting segments of travel paths. It is not a good idea to work with the standard form $y = ax + b$ of the equation of a line, since some waves will be reflected or refracted in the vertical direction. One good choice is to use the parametric equations of a line: a line passing through a point (x_0, y_0) with direction vector $v = (\cos\theta, \sin\theta)$ is the set of points

$$\{(x_0 + t\cos\theta, y_0 + t\sin\theta) \mid t \in \mathbb{R}\}.$$

For instance, waves from the epicenter travel along the lines

$$\{(1 + t\cos\theta, t\sin\theta) \mid t \in \mathbb{R}\}.$$

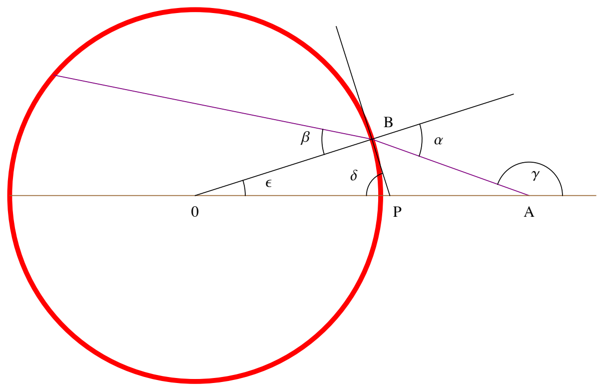


Figure 8: Computing the path of a refracted wave.

One such line intersects the middle circle at a value t such that $x(t)^2 + y(t)^2 = \frac{25}{81}$. This yields to a quadratic equation in t, which will have two positive solutions, $0 < t_1 \leq t_2$, when the line intersects the circle, i.e. $\theta \in [\theta_0, 2\pi - \theta_0]$ where $\theta_0 \approx 1.96353$, corresponding to $\tau_0 = 112^{\text{o}}$. We need to take the smallest solution, t_1, corresponding to the intersection point closest to $(1, 0)$. This gives us the point B of the middle circle, and its angular coordinate ϵ. We now need to find the equation of the refracted wave through B.

Computing a refracted wave. For this purpose, let us look at Figure 8. Our given angle is γ and we calculated ϵ after having found B. Note that $\gamma = \frac{\pi}{2} + \frac{\theta}{2}$. Indeed, consider the angle $\angle PAB$, with value $\pi - \gamma$. It is inscribed inside the large circle (which is not drawn). Its associated central angle is equal to $\pi - \theta$. Since an inscribed angle is equal to half the associated central angle, this yields $\pi - \gamma = \frac{\pi}{2} - \frac{\theta}{2}$, hence the result. Now, since PB is orthogonal to OB, then $\delta = \frac{\pi}{2} - \epsilon$. We need to calculate $\alpha = \frac{\pi}{2} - \angle PBA$. In the triangle BPA, the two other angles are $\pi - \delta$ and $\pi - \gamma$. Hence, $\angle PBA = \delta + \gamma - \pi$, yielding $\alpha = \pi + \epsilon - \gamma$. Using the law of refraction, we calculate β given that $\sin \beta = \frac{v_2}{v_1} \sin \alpha$, where v_1 (resp. v_2) is the speed of the wave outside (resp. inside) the middle circle. Then you can verify that the reflected wave makes an angle of $\pi + \epsilon - \beta$ with the horizontal right semi-axis, allowing to get its parametric equation.

Once inside the middle circle, there are three possibilities for the refracted wave:

- It can exit the middle circle without touching the small circle. You need to calculate its intersection with the middle circle as above, and find the parametric equation of the refracted wave, using the symmetries of Figure 6.
- It can be refracted when entering the small circle: the parametric equation of the refracted wave can be calculated as above.

- It can be reflected on the inner circle: we make the computation for this case below.

Computing a reflected wave. Let us look at Figure 9. We are given γ, and ϵ has been calculated after having found B. As before, $\delta = \frac{\pi}{2} - \phi$. As an exercise, you can check that $\alpha = \gamma - \epsilon - \frac{\pi}{2}$, and that the reflected wave makes an angle of $\frac{\pi}{2} + \epsilon - \alpha$ with the horizontal right semi-axis.

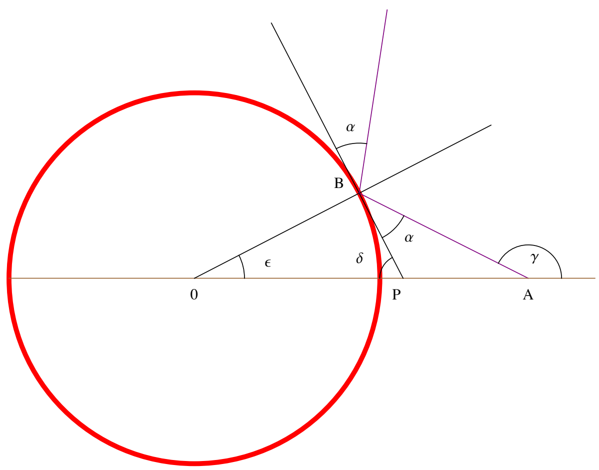


Figure 9: Computing the path of a reflected wave.

Of course this is not the end of the game and we need to iterate several of these steps. There is no special difficulty, but the calculations are a bit tedious.

4.2 Discussion of the model

We observe in Figure 7(b) that the travel times of the waves reflected by the inner core are very similar to some travel times of waves refracted inside the inner core. This is just a coincidence coming from the special values of the traveling speed of the signal in the different layers. Hence, we see that we cannot distinguish the two types of waves just from their travel time. This does not exclude that other criteria (for instance intensity of the waves) or more sophisticated methods of signal analysis would allow to distinguish them. The rough model of the inner structure of the Earth is now presented in Figure 10.

According to Inge Lehmann, she build and presented this small model to illustrate that the existence of an inner core in which the waves would travel faster would allow to explain for the waves detected in the forbidden region in the absence of an inner

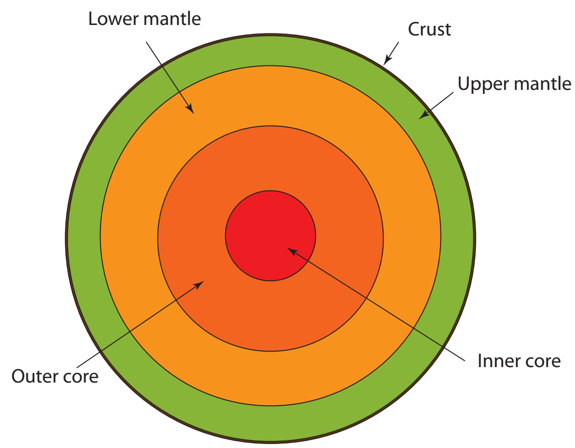


Figure 10: The inner structure of the Earth.

core. The idea was accepted by Beno Gutenberg and Charles Francis Richter (the creator of the Richter magnitude scale), seismologists at the California Institute of Technology. They placed a small inner core inside the Earth and adjusted its radius and the traveling speed of waves inside the inner core until the calculated time curves agreed with the data observed. It is also Inge Lehmann's model which led to the observation of the upper branch of refracted waves in Figure 4. Indeed, the intensity of waves in this upper branch is small and the branch went first unnoticed.

Accordingly to this description, the strategy for analyzing the inner structure of the Earth seems reasonably simple. When dealing with real data, things were not so simple. There was a reasonable network of seismic stations in Europe, but this was not the case in all areas of the world. The work we have presented relies on the hypothesis that the epicenter of the earthquake has been located precisely. But that itself requires precise observations well distributed around the epicenter, which was rarely the case. Indeed, how do you locate the epicenter?

Locating the epidenter. You have a system of four equations to solve, the four unknowns being the three coordinates, (x, y, z), of the position of the earthquake and the time of occurrence, t, of the earthquake. You observe the times when the seismic waves are registered at n different stations located not too far, so that you can assume the speed of the waves, v, to be constant along their travel paths assumed to be straight lines. From these times, you derive a system of n equations in the four unknowns (x, y, z, t). Indeed, if the ith station is located at (a_i, b_i, c_i) and registered the earthquake at time t_i, then the distance d_i from the epicenter to the station is $d_i = \sqrt{(x-a_i)^2 + (y-b_i)^2 + (z-c_i)^2}$. It is also equal to the speed multiplied by the travel time of the wave, which is $t_1 - t_i$, namely $d_i = v(t_i - t)$. This yields to a system of n quadratic equations:

$$(x-a_i)^2 + (y-b_i)^2 + (z-c_i)^2 = v^2(t_i - t)^2, \qquad i = 1, \ldots, n.$$

A minimum of four equations is necessary to obtain a finite number of solutions, in this case two solutions. More equations are needed if there is no obvious way to discard the wrong solution. Details on how to solve such systems can be found in Chapter 1 of [3], which discusses the functioning of the GPS. Of course, an additional hypothesis is that the clocks of the stations are well synchronized! But strong earthquakes usually do not happen in the middle of the European seismic stations and, according to Inge Lehmann [2], more subtle adjustments had to be made.

It took a few years before the idea of the inner core was finally accepted by the community of seismologists. Let us now cite the last sentences of [2]: *"The first results for the properties of the inner core were naturally approximate. Much has been written about it, but the last word has probably not yet been said."* Will you say one of the next words in the future?

References

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[2] Inge Lehmann, Seismology in the Days of Old, Eos, **68**(3), January 20 1987. The paper can be found at http://www.geus.dk/geuspage-dk.htm.

[3] C. Rousseau and Y. Saint-Aubin, *Mathematics and Technology*, Springer Undergraduate Texts in Mathematics and Technolgy, Springer, New York, 2008.